We are considering probability problems involving finite sample description spaces in which the descriptions are not necessarily equally likely. We are using the notion of independent and dependent events.
Consider an event $B$ being independent of an event $A$ which has positive probability. Let $A$ and $B$ be events defined on the same probability space $S$. Assume $P[A] > 0$ so that $P[B|A]$ is well defined. The event $B$ is said to be independent of the event $A$ if the conditional probability of $B$ given $A$ is equal to the unconditional probability of $B$.

$B$ is independent of $A$ if $P[B|A] = P[B]$
Independent Events

\[ P[B|A] = P[B] \] and using the property we learned previously,
\[ P[AB] = P[A]P[B|A], \]


If \( B \) is independent of \( A \) it then follows that \( A \) is independent of \( B \), therefore \( P[A|B] = P[A] \).

**Therefore we can conclude the events \( A \) and \( B \) are said to be independent if \( P[AB] = P[A]P[B] \) holds.**
Example - Independence

Consider the problem of drawing with replacement a sample of size two from an urn containing 4 white and 2 red balls. $A$ denotes the event that the first ball is white. $B$ denotes the event that the second ball is white.

Are $A$ and $B$ independent events?
Example - Independence

Now consider the same problem as before but this time without replacement

Consider the problem of drawing **without replacement** a sample of size two from an urn containing 4 white and 2 red balls. $A$ denotes the event that the first ball is white. $B$ denotes the event that the second ball is white.

Are $A$ and $B$ independent events?
The notions of independent events and of conditional probabilities may be extended to more than two events. Suppose one has three events $A$, $B$, and $C$ defined on a probability space. The events $A$, $B$, and $C$, are said to be independent if


Example - Independence

Let a ball be drawn from an urn containing four balls, numbered 1 to 4. Assume that \( S = \{1, 2, 3, 4\} \) possesses equally likely descriptions. The events \( A = \{1, 2\} \), \( B = \{1, 3\} \), and \( C = \{1, 4\} \).

Validate whether or not these descriptions satisfy pairwise independence (i.e. 
\[
)?

Now validate whether the three events are independent (i.e. 
\[
Let $A$ be an event in a sample space $S$, and let $B_1, B_2, \ldots B_n$ be mutually disjoint events whose union is $S$. Then

Bayes’ Theorem

Let the event $B_1, ... B_k$ form a partition of the space $S$ such that $P[B_j] > 0$ for $j = 1, ..., k$, and let $A$ be an event such that $P[A] > 0$.

Then for $i = 1, ..., k$,

$$P[B_i | A] = \frac{P[AB_i]}{P[A]} = \frac{P[B_i]P[A|B_i]}{P[A]} = \frac{P[B_i]P[A|B_i]}{\sum_{j=1}^{k} P[B_j]P[A|B_j]}$$
Prior and Posterior Probabilities

The probability $P[B_j]$ is called the **prior probability** because the probability that an observation will fall into a group before you collect the data.

The probability $P[B_j|A]$ is called the **posterior probability** because it is the probability of assigning observations to groups given the data.
Example - Bayes’ Theorem

A factory uses three machines, \( X \), \( Y \), and \( Z \) to produce certain items. Suppose:

- Machine \( X \) produces 50 percent of the items, of which 3% are defective.
- Machine \( Y \) produces 30 percent of the items, of which 4% are defective.
- Machine \( Z \) produces 20 percent of the items, of which 5% are defective.

Find the probability \( P[D] \) that a randomly selected item is defective, where \( D \) denotes the event that an item is defective.

Suppose a defective item is found among the output. Find the probability that it came from each of the machines (i.e. find \( P[X|D] \), \( P[Y|D] \), \( P[Z|D] \)).