COMP 1205: Foundations of CS
Linear Algebra
Linear Independence and Vector Spaces

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Topics to be covered

- Linear independence of a set of vectors
- Basis vectors and dimension of vector spaces
- Matrices represent transformation of vectors
Review: solutions of equations form weighted combination of column vectors of matrix

- Can express system of equations in two ways:

\[
\begin{align*}
2x_1 + 3x_2 &= 5, \\
-2x_1 + x_2 &= -1
\end{align*} \iff \begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}
\]

- Each equation (row) in a system of equations describes a hyperplane (line in 2D, plane in 3D)
- Intersection of hyperplanes locates solution \((x_1, x_2)\)
- \(x_1, x_2\) constructs linear combination of columns of \(A\) so that

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{-2} \\ \frac{3}{1} \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}.
\]
Are a set of vectors linearly (in)dependent?

- Equations in matrix form \( Ax = b \) for \( b = 0 \) are called **homogeneous equations**.
- Zero vector \( x = 0 \) is always a solution.
- Solution \( x \neq 0 \) is called a non-trivial solution.
- Solutions determine whether the columns of \( A \) (viewed as vectors) are **linearly (in)dependent**.
- Find \( x_1, x_2, \ldots x_N \) such that

\[
x_1 \text{col}_1(A) + x_2 \text{col}_2(A) + \cdots + x_N \text{col}_N(A) = 0.
\]

for \( N = 3 \) and

\[
(A|b) = \begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{pmatrix}.
\]

Solution determines linear dependence/independence.
Recall: row reduction steps

Task: find values of $x_1, x_2, x_3$ so that

$$x_1 \text{col}_1(A) + x_2 \text{col}_2(A) + x_3 \text{col}_3(A) = 0.$$ 

$$(A|b) = \begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{pmatrix}$$

\[
\begin{align*}
&\xrightarrow{-3\rho_1 + \rho_3} \\
&\xrightarrow{\rho_3} \\
&\xrightarrow{-2\rho_1 + \rho_2} \\
&\xrightarrow{\rho_2} \\
&\xrightarrow{-2\rho_3 + \rho_3} \\
\Rightarrow & \quad x_1 = 2x_3, \quad x_2 = -x_3, \quad x_3 = \text{any value, say } \alpha \in \mathbb{R}. 
\end{align*}
\]
Definition: linear dependence and nullspace

- We have found values $\mathbf{x} = (x_1, x_2, x_3)^T = (2, -1, 1)\alpha$, where $(\cdot)^T$ is matrix transpose so that $A\mathbf{x} = 0$. Check:

$$
\begin{pmatrix}
1 & 4 & 2 \\
2 & 5 & 1 \\
3 & 6 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= x_1 \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} + x_2 \begin{pmatrix}
4 \\
5 \\
6
\end{pmatrix} + x_3 \begin{pmatrix}
2 \\
1 \\
0
\end{pmatrix} = 0.
$$

- The vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ are linearly dependent.

- Equivalently, vector $\mathbf{x} \neq 0$ is in the kernel or nullspace of $A$, written as $\text{ker}(A)$.

- $x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_N \mathbf{v}_N = 0$ if and only if $x_i = 0$ for all $i = 1, \ldots, N$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ are linearly independent.

- Exercise: replace $a_{13} \leftarrow 1$; prove columns linearly independent.
Linear (in)dependence of a set of vectors

- Determine whether the following sets of vectors $S_1$ and $S_2$ are linearly independent. For $\mathbf{v}_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$, $S_2 = \{\mathbf{v}_1, \mathbf{v}_3\}$

- For 2 vectors, if and only if one is a multiple of the other are they linearly dependent.

Linear dependent set

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is **linearly dependent** if and only if at least one of the vectors $\mathbf{v}_k$ can be expressed as a linear combination of the others, $\mathbf{v}_k = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$, $\alpha_i \neq 0$. 
Span of a set of vectors

Spanning set

\( w \) is said to be in the span of \( V = \{v_1, \ldots, v_n\} \) (\( w \in V \)) if

\[
    w = \sum_{i=1}^{n} \alpha_i v_i = \alpha_1 v_1 + \cdots + \alpha_n v_n.
\]

- Let \( v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \) and \( v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \). Describe the set spanned by \( v_1 \) and \( v_2 \) and describe its geometry.

- Prove: \( w \in \text{Span}(v_1, v_2) \) if and only if \( \{w, v_1, v_2\} \) is linearly dependent.
Linear dependence of a number of vectors of a given dimension

Linear dependence of \( n \) \( p \)-dimensional vectors

\( n \) \( p \)-dimensional vectors \( \{v_1, \ldots, v_n\}, \ v_i \in \mathbb{R}^p \) are linearly dependent if \( p < n \).

- \( p \) corresponds to the number of variables to solve for in a set of \( n \) linear equations \( Ax = b \), \( A \) is \( n \times p \) matrix, with \( v_i \) as row \( i \).
- Example: \( v_i = (\text{feature-one}_i, \text{feature-two}_i, \ldots, \text{feature-p}_i) \).
- “Fat” matrix has more columns (variables) than rows (equations) - the solution set is underdetermined. “Thin” matrix has more rows than columns – overdetermined system of equations.
- Each row a \( p \)-dimensional hyperplane, not enough of them to intersect to a point.
- There must be at least one free variable. Equivalently, the columns of \( A \) are linearly dependent, \( Ax = 0 \) for \( x \neq 0 \).
Exercises on linear dependence

Notation: \( \mathbf{w} = \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_p \end{pmatrix} = (\mathbf{w}_1, \cdots, \mathbf{w}_p)^T. \)

1. For each of the following sets of vectors determine if they are linearly dependent.
   
   (i) \( \{\mathbf{w}_1, \mathbf{w}_2\} = \{(3, -6, -9)^T, (-4, 8, 12)^T\} \)
   
   (ii) \( \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\} = \{(1, -1)^T, (1, 3)^T, (2, 1)^T, (-1, 5)^T\} \)
   
   (iii) \( \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \{(1, 5, -3)^T, (-2, -9, 6)^T, (3, a, -9)^T\} \)

2. How many pivot columns must a \((m \times n)\) matrix have for its columns to be linearly independent, if
   
   (i) \( (m, n) = (7, 5)\)?
   
   (ii) \( (m, n) = (5, 7)\)?

3. How many pivot columns must a \((5 \times 7)\) matrix have for its columns to span \(\mathbb{R}^5\)?
Find non-trivial solutions $\mathbf{x}$ for $A\mathbf{x} = 0$ and $\mathbf{y}$ for $B\mathbf{y} = 0$; in other words, $\mathbf{x} \in \ker(A)$ and $\mathbf{y} \in \ker(B)$, where

\[
A = \begin{pmatrix}
-1 & 4 & 2 \\
1 & 1 & 3 \\
-3 & 1 & -5 \\
4 & 1 & 9
\end{pmatrix}, \quad B = \begin{pmatrix}
3 & -1 & 7 \\
-3 & 1 & -7 \\
7 & 1 & 3
\end{pmatrix}
\]

You will find that twice the first column of $A$ added to the second column yields the third. The sum of the first column of $B$ and $-4$ times the second is equal to the third column. Use these two facts to write down $\mathbf{x}$ and $\mathbf{y}$. 
Highly Recommended: Linear Algebra on YouTube 3Blue1Brown

- https://www.youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab
- Search term: “3Blue1Brown essence of linear algebra”