COMP 1205: Foundations of CS
Linear Algebra
Solving Linear Systems of Equations

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2017/8
Topics to be covered

- Gaussian Elimination – row reduction, echelon forms
- Counting solutions
- Row reduction as matrix multiplication
Chinese manuscript *Chiu-chang Suan-shu* (Nine Chapters on Arithmetic) (c. 200 BCE) had 246 practical problems including:

3 sheafs of a good crop, 2 sheafs of a mediocre crop, and 1 sheaf of a bad crop are sold for 39 dou. 2 sheafs of good, 3 mediocre, and 1 bad are sold for 34 dou; and 1 good, 2 mediocre, and 3 bad are sold for 26 dou. What is the price received for each sheaf of a good crop, each sheaf of a mediocre crop, and each sheaf of a bad crop?

Translating into mathematical notation:

\[
3x_1 + 2x_2 + 1x_3 = 39 \\
2x_1 + 3x_2 + 1x_3 = 34 \\
1x_1 + 2x_2 + 3x_3 = 26
\]

Find \(x_1, x_2\) and \(x_3\).
Solving system of equations by Gaussian Elimination

Solve for \((x_1, x_2, x_3)\):

\[
\rho_1 : \quad 3x_1 + 2x_2 + 1x_3 = 39 \\
\rho_2 : \quad 2x_1 + 3x_2 + 1x_3 = 34 \\
\rho_3 : \quad 1x_1 + 2x_2 + 3x_3 = 26
\]

- Add row 1 to \(-3\) times row 3:

\[
\begin{align*}
3x_1 + 2x_2 + 1x_3 &= 39 \\
2x_1 + 3x_2 + 1x_3 &= 34 \\
1x_1 + 2x_2 + 3x_3 &= 26 \\
\end{align*}
\]

\[
\begin{align*}
\rho_1 - 3\rho_3 & \rightarrow \quad 3x_1 + 2x_2 + 1x_3 = 39 \\
2x_1 + 3x_2 + 1x_3 & \rightarrow \quad 2x_1 + 3x_2 + 1x_3 = 34 \\
-4x_2 - 8x_3 & \rightarrow \quad -4x_2 - 8x_3 = -39 \\
\end{align*}
\]

- Add row 1 to \((-\frac{3}{2})\) times row 2:

\[
\begin{align*}
3x_1 + 2x_2 + 1x_3 &= 39 \\
-4x_2 - 8x_3 &= -39 \\
\end{align*}
\]

\[
\begin{align*}
\rho_1 - \frac{3}{2}\rho_2 & \rightarrow \quad 3x_1 + 2x_2 + 1x_3 = 39 \\
\rho_2 & \rightarrow \quad -\frac{5}{2}x_2 - \frac{1}{2}x_3 = -12 \\
\end{align*}
\]

\[
\begin{align*}
-4x_2 - 8x_3 &= -39 \\
\end{align*}
\]
Solving system of equations by Gaussian Elimination

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- We have seen:

\[
\begin{align*}
3x_1 + 2x_2 + 1x_3 &= 39 \\
2x_1 + 3x_2 + 1x_3 &= 34 \\
1x_1 + 2x_2 + 3x_3 &= 26
\end{align*}
\]

\[
\begin{align*}
3x_1 + 2x_2 + 1x_3 &= 39 \\
-\frac{5}{2}x_2 - \frac{1}{2}x_3 &= -12 \\
-4x_2 - 8x_3 &= -39
\end{align*}
\]

- Add \((-\frac{8}{5}) \) times row 2 to row 3:

\[
\begin{align*}
3x_1 + 2x_2 + 1x_3 &= 39 \\
-\frac{5}{2}x_2 - \frac{1}{2}x_3 &= -12 \rightarrow \frac{-\frac{8}{5} \rho_2 + \rho_3}{\rho_3} \\
-4x_2 - 8x_3 &= -39 \rightarrow \frac{-\frac{36}{5}x_3}{\rho_3}
\end{align*}
\]
Gaussian Elimination reduces system of equations to (reduced) row echelon form

- We have seen 3 transformations:
  \[
  \begin{align*}
  \rho_1 - 3\rho_3 & \rightarrow \rho_3 \\
  \rho_1 - \frac{3}{2}\rho_2 & \rightarrow \rho_2 \\
  -\frac{8}{5}\rho_2 + \rho_3 & \rightarrow \rho_3 
  \end{align*}
  \]

- that transform
  \[
  \begin{align*}
  3x_1 + 2x_2 + 1x_3 &= 39 \\
  2x_1 + 3x_2 + 1x_3 &= 34 \\
  1x_1 + 2x_2 + 3x_3 &= 26
  \end{align*}
  \]

  into
  \[
  \begin{align*}
  3x_1 + 2x_2 + 1x_3 &= 39 \\
  -\frac{5}{2}x_2 - \frac{1}{2}x_3 &= -12 \\
  -36x_3 &= -99
  \end{align*}
  \]

  or
  \[
  \begin{align*}
  1x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 &= 13 \\
  1x_2 + \frac{1}{5}x_3 &= \frac{24}{5} \\
  1x_3 &= \frac{11}{4}
  \end{align*}
  \]

- Solve by successive back substitution
Exercise: (Gaussian elimination:) row reduction to echelon form

Transform to reduced row echelon form

\[
\begin{align*}
    x_1 - 6x_2 + 4x_3 &= -20, \\
    2x_2 - 7x_3 &= 21, \\
    x_3 + 2x_4 &= 7, \\
    3x_3 + x_4 &= -3
\end{align*}
\]

Answer:

\[
\begin{align*}
    x_1 &= -6/5, \\
    x_2 &= 7/5, \\
    x_3 &= -13/5, \\
    x_4 &= 24/5
\end{align*}
\]
Matrix representation of systems of equations

\[ \begin{align*}
3x_1 + 2x_2 + x_3 &= 39 \\
2x_1 + 3x_2 + x_3 &= 34 \\
x_1 + 2x_2 + 3x_3 &= 26
\end{align*} \]

- \[ Ax = b \]

\[
\begin{pmatrix}
3 & 2 & 1 \\
2 & 3 & 1 \\
1 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
39 \\
34 \\
26
\end{pmatrix}
\]

- Define augmented matrix \((A|b)\):

\[
(A|b) = \begin{pmatrix}
3 & 2 & 1 & 39 \\
2 & 3 & 1 & 34 \\
1 & 2 & 3 & 26
\end{pmatrix}
\]
Solution of equations in matrix form

Row operations transform

\[
\begin{align*}
3x_1 + 2x_2 + x_3 &= 39 \\
2x_1 + 3x_2 + 1x_3 &= 34 \\
1x_1 + 2x_2 + 3x_3 &= 26
\end{align*}
\]

into

\[
\begin{align*}
1x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 &= 13 \\
1x_2 + \frac{1}{5}x_3 &= \frac{24}{5} \\
1x_3 &= \frac{11}{4}
\end{align*}
\]

Augmented matrix

\[
(A|b) = \begin{pmatrix}
3 & 2 & 1 & 39 \\
2 & 3 & 1 & 34 \\
1 & 2 & 3 & 26
\end{pmatrix}
\]

is transformed into

\[
\begin{pmatrix}
1 & \frac{2}{3} & \frac{1}{3} & 13 \\
0 & 1 & \frac{1}{5} & \frac{24}{5} \\
0 & 0 & 1 & \frac{11}{4}
\end{pmatrix}
\]
Matrix representation of systems of equations - example 2

- $Ax = b$

Define augmented matrix $(A|b)$:

$$
\begin{pmatrix}
1 & -6 & 4 & 0 & -20 \\
0 & 2 & -7 & 0 & 21 \\
0 & 0 & 1 & 2 & 7 \\
0 & 0 & 3 & 1 & -3
\end{pmatrix}
$$
Solution of equations in matrix form – example 2

Row operations transform

\[
\begin{align*}
  x_1 - 6x_2 + 4x_3 &= -20, \\
  2x_2 - 7x_3 &= 21, \\
  x_3 + 2x_4 &= 7, \\
  3x_3 + x_4 &= -3
\end{align*}
\]

into

\[
\begin{align*}
  x_1 &= -6/5, \\
  x_2 &= 7/5, \\
  x_3 &= -13/5, \\
  x_4 &= 24/5
\end{align*}
\]

Augmented matrix

\[
(A|b) = \begin{pmatrix}
  1 & -6 & 4 & 0 & -20 \\
  0 & 2 & -7 & 0 & 21 \\
  0 & 0 & 1 & 2 & 7 \\
  0 & 0 & 3 & 1 & -3
\end{pmatrix}
\]

is transformed into

\[
\begin{pmatrix}
  1 & 0 & 0 & 0 & -6/5 \\
  0 & 1 & 0 & 0 & 7/5 \\
  0 & 0 & 1 & 0 & -13/5 \\
  0 & 0 & 0 & 1 & 24/5
\end{pmatrix}
\]
(Reduced) Row Echelon Form: Definition

- A rectangular matrix is in **row echelon form** if each leading (non-zero) entry of a row (■, called **pivot**) is to the right of the leading entry of the row above. (□ is any number, including 0.)

- A rectangular matrix is in **reduced row echelon form** if the leading entry of each row is 1, and is the only non-zero element in its column.
Row operators, row equivalence of matrices and solutions of linear equations

The row operations:

1. Interchange 2 rows: \( \rho_i \leftrightarrow \rho_j \)
2. Multiply all entries in a row by a non-zero constant: \( \rho_i \leftarrow k\rho_i \)
3. Replace a row by the sum of itself and a multiple of another row: \( \rho_i \leftarrow \rho_i + k\rho_j \)

Two matrices are **row equivalent** if they are transformed into each other by row operations.

If the augmented matrices of two systems of equations are row equivalent, then the two systems of equations have the same solutions.
Existence and uniqueness of solutions of linear equations

1. Does there exist at least one solution? Equivalently, is the system of equations consistent?

2. If a solution exists, is it unique?
Exercise: find solutions of linear equations

If you are lucky, you should find the solutions \((x_1, x_2, x_3)\) to the following set of equations. Find the **row echelon form** for the system of equations:

\[
\begin{align*}
2x_1 - 3x_2 + 4x_3 &= 2 \\
3x_1 - 5x_2 + x_3 &= 1 \\
-x_1 + 3x_2 + cx_3 &= -2
\end{align*}
\]

How can you get unlucky? What prevents the reduction to the **reduced** row echelon form?

**Hint:** the sequence of row operations is:

\[
\frac{\rho_1 + 2\rho_3}{\rho_3} \quad \frac{-\frac{3}{2}\rho_1 + \rho_2}{\rho_2} \quad \frac{6\rho_2 + \rho_3}{\rho_3}
\]
Exercise: existence of solutions of linear equations

Show that the augmented matrix \((Ab)\) of \(Ax = b\) can be row-reduced as follows:

\[
(\begin{array}{cccc}
9 & -15 & 3 & -7 \\
3 & -5 & 1 & 1 \\
-1 & 3 & 1 & -2 \\
\end{array}) \rightarrow \left( \begin{array}{cccc}
1 & 0 & 2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right)
\]

Solve for \(x = (x_1, x_2, x_3)\). How many solutions exist?

Another system of equations in unknowns \((x_1, x_2, x_3)\) is

\[
\begin{align*}
2x_1 + x_2 + 4x_3 &= 7, \\
2x_1 - 5x_2 + x_3 &= -5, \\
-4x_1 + 4x_2 - 5x_3 &= -2
\end{align*}
\]

Solve for \((x_1, x_2, x_3)\).
Exercise: Balancing chemical reactions

- \( x_1 \text{C}_3\text{H}_8 + x_2 \text{O}_2 \rightarrow x_3 \text{CO}_2 + x_4 \text{H}_2\text{O}. \) Represent the compounds as

\[
\begin{align*}
\text{C}_3\text{H}_8 : & \begin{pmatrix} 3 \\ 8 \\ 0 \end{pmatrix}, \\
\text{O}_2 : & \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \\
\text{CO}_2 : & \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \\
\text{H}_2\text{O} : & \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.
\end{align*}
\]

- Balancing reactions: find \( x_1, x_2, x_3, x_4 \) such that

\[
\begin{pmatrix} 3 \\ 8 \\ 0 \end{pmatrix}x_1 + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}x_2 - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}x_3 - \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}x_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

- The system of equations in unknowns \((x_1, x_2, x_3, x_4)\) is

\[
\begin{align*}
3x_1 + 0x_2 - 1x_3 - 0x_4 &= 0, \\
8x_1 - 0x_2 - 0x_3 - 2x_4 &= 0, \\
0x_1 + 2x_2 - 2x_3 - 1x_4 &= 0.
\end{align*}
\]

- How many solutions can you find?

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Summary: Solutions to linear equation set using Gaussian elimination

Perform row reduction (Gaussian elimination) to obtain row echelon form:

- If it has a contradictory row \((0 \cdots 0x), x \neq 0\): no solution.
- If no contradictory rows, and each row has its variable as pivot (leading variable): unique solution.
- If no contradictory rows, and even one row has no pivot, infinitely many solutions.
Row replacement as matrix multiplication

- Extract 3rd row from matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and place it in the 2nd row by finding $E_{23}$ so that

$$E_{23}A = \begin{pmatrix} 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 \end{pmatrix}$$

- Verify:

$$E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
Exercise: row replacement

For $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $E_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, what is $E_{32}A$?

The $(i, j)^{th}$ element of matrix $E_{rs}$ that places row $s$ where row $r$ used to be and sets all other elements to zero is

$$(E_{rs})_{ij} = \delta_{ri} \delta_{sj} \text{ where } \delta_{pq} = \begin{cases} 1 & p = q \\ 0 & \text{otherwise.} \end{cases}$$
Row operation $\rho_k \mapsto \rho_k + \chi \rho_i$: add $\chi$ times row $i$ to row $k$

- Copy matrix $A$ by multiplying with the identity matrix $I$ and then add $\chi E_{32}$. So $(I + \chi E_{32})A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \chi & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
\chi a_{21} + a_{31} & \chi a_{22} + a_{32} & \chi a_{23} + a_{33}
\end{pmatrix}
\]
Row interchange as matrix multiplication

- Interchange 2 rows: $\rho_1 \leftrightarrow \rho_2 : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
5 \\
3 \\
4 \\
2 \\
6 \\
\end{pmatrix}
\]

- The $(i, j)^{th}$ element of matrix $P_{rs}$ that permutes (interchanges) rows $r$ and $s$ is written as

$$
(P_{rs})_{ij} = \begin{cases} 
1 & i = j \text{ unless } i = j = r, \text{ or } i = j = s \\
1 & (i, j) = (r, s) \text{ and } (i, j) = (s, r) \\
0 & \text{otherwise.}
\end{cases}
$$
Summary: Reduction to echelon form by matrix multiplication

1. Interchange 2 rows of matrix $A$: $\rho_i \leftrightarrow \rho_j$ achieved by $P_{rs}A$.
2. Multiply all entries in a row of $A$ by a non-zero constant: $\rho_i \leftarrow \chi \rho_i$ is achieved by diagonal matrix $\mathbb{I} + \chi \mathbf{E}_{ii}$.
3. Replace a row of $A$ by the sum of itself and a multiple of another row: $\rho_i \leftarrow \rho_i + \chi \rho_j$ is achieved by $\mathbb{I} + \chi \mathbf{E}_{ij}$.
4. Notation: observe that operations $\Theta_{ij}$ act as $i \leftarrow j$. 
Many books in the library

Free textbooks:

2. http://joshua.smcvt.edu/linearalgebra/
3. http://linear.ups.edu/
5. http://cseweb.ucsd.edu/~gill/CILASite/