COMP1215: Combinatorics — art of counting

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On the Combinatorial Art
Gottfried Leibniz (1666)

"Can you do Addition?" the White Queen said. "What's one and one and one and one and one and one and one and one and one and one?"
"I don't know," said Alice. "I lost count."
"She can't do Addition," the Red Queen interrupted.

Lewis Carroll
Topics

• Principle of inclusion and exclusion

• Pigeonhole principle

• Recurrences and proof by induction

• Counting sequences: permutations and combinations (next week)
Principle of inclusion and exclusion

\[ |A \cup B| = |A| + |B| - |A \cap B| \]
Principle of inclusion and exclusion

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

• How many positive integers less than or equal to 150 are relatively prime to 70?
• \(70 = 2 \times 5 \times 7\)
• \(A, B, C\) multiples of 2, 5, 7
Pigeonhole principle

A human has a maximum of around 200,000 hairs on her head. Prove that at least 2 people in London have exactly the same number of hairs on their head.

$n = 10$ pigeons, $m = 9$ holes. Since $10 > 9$, at least one hole has more than one pigeon. Wikipedia

Wikipedia
Pigeonhole principle

If \(|A| > |B|\), every function from \(A\) to \(B\) maps at least 2 distinct elements of \(A\) to the same element of \(B\). For \(f : A \rightarrow B\) \(\exists x_1, x_2 \in A, x_1 \neq x_2\), such that \(f(x_1) = f(x_2)\).

To use this principle, find:

1. set \(A\) (pigeons)
2. set \(B\) (pigeonholes)
3. function \(f\) (assign pigeons to pigeonholes)
Pigeonhole principle exercises

1. For any \((n+1)\) numbers taken from set of positive integers \(\{1, 2, \ldots, 2n\}\) a pair will have no factors in common.
   - **Pigeonholes**: \(\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}\) (n in number)
   - **Pigeons**: any \((n+1)\) integers between 1 and 2n.

2. From 10 distinct 2 digit numbers we can always find two disjoint non-empty subsets with the same sum.
   - **Pigeonholes**: Sum of ten numbers \(\leq 990\)
   - **Pigeons**: subsets, number \(2^{10}=1024\)
Fibonacci series and recursion

\[ |A| = \text{cardinality of set } A = \#\{a | a \in A\} \]

\[ \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots \} \]

Recurrence relation:

\[ f_n = f_{n-1} + f_{n-2} \]

\[
\begin{pmatrix}
    f_n \\
    f_{n-1}
\end{pmatrix}
= 
\begin{pmatrix}
    1 & 1 \\
    1 & 0
\end{pmatrix}
\begin{pmatrix}
    f_{n-1} \\
    f_{n-2}
\end{pmatrix}
\]
Some relations between Fibonacci numbers

1. \( S_n = \sum_{i=1}^{n} f_n = f_{n+2} - 1. \)
2. \( f_{n-1} f_{n+1} - f_n^2 = (-1)^n \)
3. \( f_{n+1}^2 + f_n^2 = f_{2n+1} \)
4. \( f_{n+m} = f_{n-1} f_m + f_n f_{m+1} \)
5. \( f_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} \), where \( \phi = \frac{(1+\sqrt{5})}{2} \)
Inductive proofs and recursion \( \forall n : P(n) \)

- First, prove \( P(1) \) is true (basis of proof)
- If \( P(0), P(1), \ldots, P(n) \) true, (induction hypothesis)
- then \( P(n + 1) \) true
- \( [P(0) \land P(1) \land \cdots \land P(n)] \rightarrow P(n + 1) \)
Exercises on proofs by induction

1. Sum of first $n$ odd numbers is $n^2$
2. Sum of first $n$ natural numbers is $\frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} (i(i + 1))^{-1} = \frac{n}{n+1}$
4. $\prod_{i=2}^{n} (1 - \frac{1}{i^2}) = \frac{n+1}{2n}$
5. $n! > 2^n$
6. $(1 + x)^n \geq 1 + nx$
Proofs by induction for Fibonacci numbers

Define

\[
Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix}
\]

1. Prove by induction

\[
Q^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}
\]

2. Using \( \det(AB) = \det(A)\det(B) \),

show that \( f_{n+1}f_{n-1} - f_n^2 = (-1)^n \)

3. Using \( Q^{2n+1} = Q^nQ^{n+1} \),

show that \( f_{n+1}^2 + f_n^2 = f_{2n+1} \)
Ratio of successive Fibonacci numbers converges to largest eigenvalue of $Q$

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http://mathworld.wolfram.com/GoldenRatio.html
Plenty of stuff out there, use a search engine …


http://mathworld.wolfram.com/FibonacciQ-Matrix.html