RELATIONS
RELATIONS BETWEEN SETS

• Recall that given two sets \( X \) and \( Y \), their **cartesian product** \( X \times Y \) is the set of all ordered pairs \((x, y)\) with \( x \in X \) and \( y \in Y \).

• A (binary) relation \( R \) from \( X \) to \( Y \) is **some** set of ordered pairs \((x, y)\) with \( x \in X \) and \( y \in Y \)

  • in other words, a subset of the cartesian product \( X \times Y \)

  • ie \( R \subseteq X \times Y \)

• We will use \( R, S, T \) to denote relations

• Q. Given finite sets \( X \) and \( Y \), how many relations are there from \( X \) to \( Y \)?
EXAMPLES OF RELATIONS

• When $X = Y$, instead of saying that $R$ is a relation from $X$ to $X$ we usually say that $R$ is a relation on $X$

Examples of relations:
• The full relation $X \times Y$, the empty relation $\emptyset \subseteq X \times Y$
• There is a relation on any set $X$ called the identity relation
  • $I_X = \{ (x, x) \mid x \in X \}$
EXAMPLE (DIGRAPHS)

• A directed graph (digraph) with at most one edge from one vertex to another can be formalised as a binary relation on its set of vertices

\{(0,0), (0,1), (0,2), (1,2)\}
COMPOSITION OF RELATIONS

• if \( R \) is a relation from \( X \) to \( Y \) and \( S \) is a relation from \( Y \) to \( Z \) then \( R ; S \) is a relation from \( X \) to \( Z \) defined
  • \((x, z) \in R ; S \) if there exists \( y \in Y \) s.t. (such that) \((x, y) \in R \) and \((y, z) \in S \)
• \( R ; S \) is sometimes written \( S \circ R \)
• composition is associative

• **Lemma:** \( I_X ; R = R = R ; I_Y \) (prove it!)
1. Represent the digraph above as a relation $R$ on the set $\{0, 1, 2\}$
2. What is the cardinality $|R|$?
3. Draw $R;R$ as a graph.
FUNCTIONS ARE SPECIAL RELATIONS

- Any function $f : X \rightarrow Y$ is a special kind of relation from $X$ to $Y$ that satisfies
  - for all $x \in X$ there exists a unique $y \in Y$ s. t. $(x, y) \in f$
  - i.e. for all $x \in X$ there exists $y \in Y$ such that $(x, y) \in f$
  - if $(x, y) \in f$ and $(x, z) \in f$ then $y = z$
- Given a function $f$, when we want to emphasise that we are talking about the relation, we call it the graph of $f$
FACT

• Composition of functions is a special case of composition of relations

that is, since functions are special kinds of relations, composing them as relations has the same effect as composing them as functions!
Question

Suppose that a relation $R$ from $X$ to $Y$ has an inverse $S$ from $Y$ to $S$. That is,

$$R \circ S = \text{id}_X \quad \text{and} \quad S \circ R = \text{id}_Y.$$ 

Prove that $R$ is the graph of a bijective function.
EQUIVALENCE RELATIONS
EQUIVALENCE RELATIONS

• An equivalence relation \( \sim \) on \( X \) (\( \sim \subseteq X \times X \)) is a special relation that satisfies the following three axioms:

  • reflexivity: \( \forall x \in X. x \sim x \)
  • symmetry: \( \forall x, y \in X. \text{ if } x \sim y \text{ then } y \sim x \)
  • transitivity: \( \forall x, y, z \in X. \text{ if } x \sim y \text{ and } y \sim z \text{ then } x \sim z \)
EXAMPLES AND NON EXAMPLES

• Let $P =$ set of people in this room. Let
  
  $R = \{ (x,y) \mid x \in P \text{ and } y \in P \text{ have the same first name } \} -$ is $R$ an equivalence relation?

• Let $C =$ set of cities in the UK
  
  $S = \{ (x,y) \mid x \in C \text{ and } y \in C \text{ are less than 50 miles from each other } \} -$ is $S$ an equivalence relation?

• Let $\mathbb{N}_{>1}$ be the natural numbers greater than 1
  
  $P = \{ (k,m) \mid k \in \mathbb{N}_{>1} \text{ and } m \in \mathbb{N}_{>1} \text{ have the same number of primes in their prime factorisation}\} -$ is $P$ an equivalence relation?
EQUIVALENCE CLASSES

• If \( \sim \) is an equivalence relation on \( X \) and \( a \in X \) then the **equivalence class** of \( a \) is the set
  \[
  [a] = \{ x \mid x \in X \text{ and } x \sim a \}
  \]

• notice that reflexivity of \( \sim \) implies that \( a \in [a] \)

• \( X/\sim = \{ [x] \mid x \in X \} \) is the set of equivalence classes
Let \( \mathbb{Z} \) be the set of integers. Let \( \sim \subseteq \mathbb{Z} \times \mathbb{Z} \) be defined \( x \sim y \) whenever there exists an integer \( k \) such that \( x = y + 4k \)

1. Show that \( \sim \) is an equivalence relation

2. What is the cardinality of \( \mathbb{Z}/\sim \)?
PARTITIONS

• A collection $X_1, \ldots, X_n$ of subsets of $X$ is a **partition** of a set $X$ when

• for all $1 \leq i, j \leq n$, $i \neq j$ implies that $X_i \cap X_j = \emptyset$
THEOREM

• (Theorem 3.4.8 on pg. 399) If $\sim$ is an equivalence relation on $X$ then the set of equivalence classes $X/\sim=\{ [x] | x \in X \}$ is a partition of $X$

• Proof:
  • for all $x \in X$ we have $x \in [x]$ so every element is in some equivalence class
  • if $x \in [a]$ and $x \in [b]$ then $x \sim a$ and $x \sim b$ so $a \sim b$ (why?) - hence $[a] \subseteq [b]$ and $[b] \subseteq [a]$, so $[a]=[b]$
QUOTIENTS

• If ~ is an equivalence relation on X then the set $X/\sim$ of equivalence classes is sometimes called the quotient of X with respect to ~

Q. There is a function $\varphi : X \rightarrow X/\sim$ defined $\varphi(x) = [x]$. When is it surjective? When is it injective?
PARTIAL ORDERS
ORDERS

• We all know that $0 \leq 1$, $4 \leq 56$, and $3 \leq \pi \leq 4$

• The order symbol is actually a relation

\[
\leq_N = \{ (p,q) \mid p,q \in \mathbb{N}, \exists r \in \mathbb{N}, q=p+r \}
\]

\[
\leq_R = \{ (x,y) \mid x, y \in \mathbb{R}, \exists x \in \mathbb{R}^+, y=x+r \}
\]
PARTIAL ORDERS

A relation \( \leq \subseteq X \times X \) is a partial order when it is

- reflexive: \( \forall x \in X. x \leq x \)
- transitive: \( \forall x,y,z \in X. x \leq y \) and \( y \leq z \) imply \( x \leq z \)
- anti-symmetric: if \( \forall x,y. \) if \( x \leq y \) and \( y \leq x \) then \( x = y \)

\( \leq_n \) and \( \leq_R \) are examples of partial orders. Actually, they are total orders, because they satisfy the additional property that for all \( x, y \) either \( x \leq y \) or \( y \leq x \).
EXAMPLE OF PARTIAL ORDERS

• Let $\leq$ be defined on the set of all people as follows

• $p \leq q$ whenever either $p = q$ or $q$ is an ancestor (parent, grandparent, great grandparent, …) of $p$

• is this a total order?