COMP1215 - FOUNDATIONS OF COMPUTER SCIENCE
Lecture 9 - Natural deduction for propositional logic
FORMAL PROOF SYSTEMS

• Last time we discussed formal proof systems in general, together with the concepts of soundness and completeness.

• In this lecture we will examine one such proof system, natural deduction, in detail

• natural deduction is sound and complete for propositional logic
FORMAL PROOFS

- **Recall**: syntax trees
  - internal nodes labelled with operations (¬, ∧, ∨, →)
  - leaves labelled with propositional variables (p, q, r, ...) or constant (⊥)

- Formal proofs in natural deduction (aka **derivations** or **proof trees**) are different kinds of trees
  - nodes labelled with formulas
  - labels of each parent & its children correspond to particular proof rule
  - leaves are assumptions traditionally drawn with leaves (assumptions) at the top and the root (conclusion) at the bottom
• A proof can be seen as a tree where
  • leaves are **assumptions**
  • the root is the proved formula
  • the internal nodes are determined by applications of **proof rules**

• In natural deduction, proof rules naturally fall into two classes
  • **introduction rules** - allow the introduction of a logical connective
  • **elimination rules** - allow the elimination of a logical connective
RULES OF NATURAL DEDUCTION
RULES FOR CONJUNCTION

∧-introduction

\[ \varphi \quad \psi \]

\[ \varphi \land \psi \]

∧-elimination

\[ \varphi \land \psi \]

\[ \varphi \]

\[ \psi \]

• one introduction rule, two elimination rules
RULES FOR IMPLICATION

→-introduction

\[
[\varphi] \\
\vdots \\
\psi \\
\hline 
\varphi \rightarrow \psi
\]

→-elimination
(modus ponens)

\[
\varphi \quad \varphi \rightarrow \psi \\
\hline 
\psi
\]

• one introduction rule, one elimination rule
A proof in natural deduction of $\chi$ with assumptions $\varphi \land \psi$, $\varphi \to (\psi \to \chi)$

\[ \varphi \land \psi \quad \land E \quad \psi \quad \land E \quad \varphi \to (\psi \to \chi) \quad \to E \quad \psi \to \chi \quad \to E \quad \chi \]

no cancelling of hypotheses
DISCHARGING ASSUMPTIONS

∧-introduction

\[ \varphi \quad \psi \]
\[ \underline{\varphi \land \psi} \]

→-introduction

\[ \begin{array}{c}
[\varphi] \\
\vdots \\
\psi \\
\end{array} \]
\[ \varphi \to \psi \]
• Give a natural deduction proof of $\chi \rightarrow \varphi \land \psi$, using assumptions $\chi \rightarrow \varphi$ and $\chi \rightarrow \psi$. 
RULES FOR NEGATION

¬-introduction

\[
\begin{array}{c}
[\varphi] \\
\vdots \\
\bot \\
\hline \\
\neg \varphi
\end{array}
\]

¬-elimination

\[
\begin{array}{c}
\varphi \\
\hline \\
\bot
\end{array}
\]

\[
\begin{array}{c}

\varphi \\
\hline \\
\neg \varphi
\end{array}
\]

- sometimes \( \neg \varphi \) is understood as shorthand for \( \varphi \to \bot \)

\[
\begin{array}{c}
[\varphi] \\
\vdots \\
\psi \\
\hline \\
\varphi \to \psi
\end{array}
\]

\[
\begin{array}{c}
\varphi \\
\hline \\
\psi
\end{array}
\]

\[
\begin{array}{c}
\varphi \\
\hline \\
\varphi \to \psi
\end{array}
\]

\[
\begin{array}{c}
\psi
\end{array}
\]
RULES FOR DISJUNCTION

- two introduction rules, one elimination rule
Proof of $\chi$ with assumptions $\psi$ and $(\phi \land \neg \psi) \lor (\phi \land \chi)$
• Give a natural deduction proof of $\neg\psi \rightarrow \neg\phi$, using assumption $\phi \rightarrow \psi$
RULES FOR ‘FALSE’

\[
\frac{\bot}{\varphi}
\]

\[
\frac{\neg\varphi}{\bot}
\]

• Note: the second rule is non-constructive: *intuitionistic* logic rules it out.
The two rules have the same power.

Using these rules can result in proofs where you prove that something exists, but don’t know what it is.
• **Theorem.** There exist irrational numbers $p, q$ such that $p^q$ is rational.

• **Proof.** Consider $\sqrt{2}$ and $(\sqrt{2})^{\sqrt{2}}$. It is well-known that $\sqrt{2}$ is not rational.

Now either $(\sqrt{2})^{\sqrt{2}}$ rational or it is not (excluded middle!). If it is rational then we are finished: let $p=q=\sqrt{2}$. If not, let $p=(\sqrt{2})^{\sqrt{2}}$ and $q=\sqrt{2}$. Then $p^q$ is rational since

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2} \times \sqrt{2}} = (\sqrt{2})^2 = 2.$$

So we proved what we wanted, but we still don’t know what $p$ and $q$ are!
This set of rules will be given to you in the exam.
PROPERTIES OF NATURAL DEDUCTION
• Recall that soundness is the statement

\[ \Gamma \vdash \varphi \implies \Gamma \vDash \varphi \]

Natural deduction is sound. Invent a proof rule that is not sound.
PROVING SOUNDNESS

• Proof (sketch): Induction on the length of the natural deduction proof
  • eg. $\lor$-elimination

\[
\begin{array}{c}
\varphi \\
\psi \\
\vdash \\
\chi & \chi & \varphi \lor \psi \\
\hline
\chi
\end{array}
\]

• assume that an assignment of truth values makes $\varphi \lor \psi$ true
• then one of $\varphi$ or $\psi$ is true
• in each case, use inductive hypothesis to get $\chi$ true
COMPLETENESS

• Completeness of a set of proof rules means that there are “enough proof rules to prove all theorems, i.e. all true statements”

• **Completeness Theorem**: \( \Gamma \vdash \phi \) implies \( \Gamma \vdash \phi \)

• Proof: out of scope, see Truss section 7.1.5