1. Let \( A \overset{\text{def}}{=} \{0, -1, 5\} \) and \( B \overset{\text{def}}{=} \{1, 3, 0, 3\} \). List all the elements of the following sets using the standard curly brace notation. For each set give its cardinality:

a) \( A \cup B \);

**Solution.** First let us write out the elements of \( A \) and \( B \) in order without repetition (note: although the order is not important, writing out \( A \) and \( B \) this way means that we can avoid making silly mistakes). So \( A = \{-1, 0, 5\} \) and \( B = \{0, 1, 3\} \). We have \( |A| = |B| = 3 \).

Now \( A \cup B = \{-1, 0, 1, 3, 5\} \) and so \( |A \cup B| = 5 \).

b) \( A \cap B \);

**Solution.** \( A \cap B = \{0\} \) and \( |A \cap B| = 1 \).

c) \( A \times B \);

**Solution.** We have

\[
A \times B = \{-1, 0, 5\} \times \{0, 1, 3\} = \{(-1, 0), (-1, 1), (-1, 3), (0, 0), (0, 1), (0, 3), (5, 0), (5, 1), (5, 3)\}.
\]

and so \( |A \times B| = 9 \).

d) \( A - B \);

**Solution.** All the elements in \( A \) that are not in \( B \), so \( \{-1, 5\} \).

e) \( (A \times \{0\}) \cup (B \times \{1\}) \);

**Solution.** First note that

\[
A \times \{0\} = \{(-1,0), (0,0), (5,0)\}
\]

and
\[ B \times \{1\} = \{(0, 1), (1, 1), (3, 1)\}. \]

Then \((A \times \{0\}) \cup (B \times \{1\}) = \{(-1, 0), (0, 0), (5, 0), (0, 1), (1, 1), (3, 1)\}\)
and \(|(A \times \{0\}) \cup (B \times \{1\})| = 6.\)

2. Find a predicate \(\varphi\) to express the following sets of integers using the notation \(\{x \mid \varphi(x)\}\).

   a) The set of all natural numbers divisible by 7;

   **Solution.** We could write simply

   \[ \{x \mid x \text{ is a natural number and is divisible by 7} \} \]

   but it is customary to reduce the amount of English phrases and use more precise mathematical language. After all, perhaps someone does not know the meaning of “is divisible by” in English. So we could write

   \[ \{x \mid x \text{ is a natural number and there exists a natural number } y \text{ such that } x = 7y \} \]

   We know that the set of natural numbers is conventionally denoted by \(\mathbb{N}\), so we could shorten the above to:

   \[ \{x \mid x \in \mathbb{N} \text{ and there exists } y \in \mathbb{N} \text{ such that } x = 7y \} \]

   This is pretty good. We could make it even shorter using the \(\exists\) shorthand (stands for “there exists”). Also, mathematicians often write s.t. for “such that” (because “such that” is used very often in maths). So we get:

   \[ \{x \mid x \in \mathbb{N} \text{ and } \exists y \in \mathbb{N} \text{ s.t. } x = 7y \}. \]

   Note that for many authors the integers are the “default” universe so instead of writing \(x \in \mathbb{N}\) they write simply \(x \geq 0\). Such an author would write:

   \[ \{x \mid x \geq 0 \text{ and } \exists y \geq 0 \text{ s.t. } x = 7y \}. \]

   b) \(\{x \mid x \geq 0 \text{ and } x < 1000\} \cap \{x \mid x > -10 \text{ and } x \leq 10\};\)

   **Solution.** \(\{x \mid x \geq 0 \text{ and } x \leq 10\}\)
c) \( \{0,1\} \times \{x \mid x \geq 0 \text{ and } x \leq 5\} \);

**Solution.** \( \{(x,y) \mid x \in \{0,1\}, y \geq 0 \text{ and } y \leq 5\} \)

3. Prove the following. You can use Venn diagrams for inspiration.

a) \( X \cup (Y \cup Z) = (X \cup Y) \cup Z \);

**Solution.** Remember that, for sets \( X \) and \( Y \), to show that \( X = Y 
\)

We will begin by showing that \( X \cup (Y \cup Z) \subseteq (X \cup Y) \cup Z \). So assume \( w \in X \cup (Y \cup Z) \). Then by definition of ‘\( \cup \)’ either \( w \in X \) or \( w \in Y \cup Z \). If \( w \in X \) then \( w \in X \cup Y \) and so \( w \in (X \cup Y) \cup Z \). The other possibility is \( w \in Y \cup Z \), which by definition means that either \( w \in Y \) or \( w \in Z \). If \( w \in Y \) then \( w \in X \cup Y \) and so \( w \in (X \cup Y) \cup Z \). Similarly, if \( w \in Z \) then \( w \in (X \cup Y) \cup Z \) and we have shown that \( X \cup (Y \cup Z) \subseteq (X \cup Y) \cup Z \).

The proof that \( (X \cup Y) \cup Z \subseteq X \cup (Y \cup Z) \) is similar and is left for you as an exercise.

b) \( X \cap (Y \cap Z) = (X \cap Y) \cap Z \);

**Solution.** Assume that \( w \in X \cap (Y \cap Z) \). Then by definition of ‘\( \cap \)’ we have \( w \in X \) and \( w \in Y \cap Z \). The latter means that \( w \in Y \) and \( w \in Z \). Since \( w \in X \) and \( w \in Y \) we have \( w \in X \cap Y \). Since also \( w \in Z \) we have \( w \in (X \cap Y) \cap Z \). thus \( X \cap (Y \cap Z) \subseteq (X \cap Y) \cap Z \).

The direction \( (X \cap Y) \cap Z \subseteq X \cap (Y \cap Z) \) is similar and is left for you as an exercise.

c) \( X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \);

**Solution.** First we show that \( X \cap (Y \cup Z) \subseteq (X \cap Y) \cup (X \cap Z) \).

Suppose that \( w \in X \cap (Y \cup Z) \). Then \( w \in X \) and \( w \in Y \cup Z \). But \( w \in Y \cup Z \) means that \( w \in Y \) or \( w \in Z \). If \( w \in Y \) then \( w \in X \cap Y \)

and so \( w \in (X \cap Y) \cup (X \cap Z) \). If \( w \in Z \) then \( w \in X \cap Z \) and so \( w \in (X \cap Y) \cup (X \cap Z) \).

Now we show that \( (X \cap Y) \cup (X \cap Z) \subseteq X \cap (Y \cup Z) \). If \( w \in (X \cap Y) \cup (X \cap Z) \) then \( w \in X \cap Y \) or \( w \in X \cap Z \). Then if \( w \in X \cap Y \) then \( w \in X \) and \( w \in Y \). So \( w \in Y \cup Z \) and thus \( w \in X \cap (Y \cup Z) \).

Otherwise if \( w \in X \cap Z \) then \( w \in X \) and \( w \in Z \), so \( w \in Y \cup Z \) and so \( w \in X \cap (Y \cup Z) \).

d) \( X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z) \);
**Solution.** Similar to the previous proof. Left as an exercise. It is also a special case of the next exercise.

e) \( X \cup (Y_1 \cap \cdots \cap Y_k) = (X \cup Y_1) \cap \cdots \cap (X \cup Y_k) \) for any \( k > 0 \) and sets \( Y_1, \ldots, Y_k \);

**Solution.** We can be more efficient with notation by writing:

\[ X \cup \bigcap_{1 \leq i \leq k} Y_i = \bigcap_{1 \leq i \leq k} X \cup Y_i. \]

First we show that \( X \cup \bigcap_{1 \leq i \leq k} Y_i \subseteq \bigcap_{1 \leq i \leq k} X \cup Y_i \). If \( w \in X \cup \bigcap_{1 \leq i \leq k} Y_i \) then \( w \in X \) or \( w \in \bigcap_{1 \leq i \leq k} Y_i \). If \( w \in X \) then for all \( i \) we have \( w \in X \cup Y_i \) and so \( w \in \bigcap_{1 \leq i \leq k} X \cup Y_i \).

Now to show \( \bigcap_{1 \leq i \leq k} X \cup Y_i \subseteq X \cup \bigcap_{1 \leq i \leq k} Y_i \) suppose that \( w \in \bigcap_{1 \leq i \leq k} X \cup Y_i \). Then for all \( i \) we have \( w \in X \cup Y_i \). Now \( w \) is either in \( X \) or not in \( X \). If \( w \in X \) then \( w \in X \cup Y_i \) and we are finished. If \( w \notin X \) then for each \( i \) we must have \( w \in Y_i \) (why?) so \( w \in \bigcap_{1 \leq i \leq k} Y_i \) and so \( w \in X \cup \bigcap_{1 \leq i \leq k} Y_i \).

4. Suppose \( X, Y \) and \( Z \) are sets. Does \( X \times (Y + Z) = X \times Y + X \times Z \)? If \( X, Y \) and \( Z \) are finite, what can we say about the cardinalities of \( X \times (Y + Z) \) and \( X \times Y + X \times Z \)?

**Solution.**

\[ X \times (Y + Z) = \{(x, (y, 0)) \mid x \in X, y \in Y\} \cup \{(x, (z, 1)) \mid x \in X, y \in Y\} \]

\[ X \times Y + Y \times Z = \{(x, (y, 0)) \mid x \in X, y \in Y\} \cup \{(x, (z), 1) \mid x \in X, z \in Z\} \]

These two are clearly not equal as sets because they have different elements: for example, given \( x \in X, y \in Y \), by the definition of ordered pairs it is not the case that \( (x, (y, 0)) = ((x, y), 0) \). On the other hand the cardinalities are the same, one can show this by exhibiting the obvious bijection between the two sets or a simple calculation:

\[ |X \times (Y + Z)| = |X||Y + Z| = |X||(|Y| + |Z|) = |X||Y| + |X||Z| = |X \times Y| + |X \times Z| = |X \times Y + X \times Z|. \]

5. Suppose that \( X \) and \( Y \) are sets and \( X \times Y \) is their cartesian product. There are two functions known as the projections: \( \pi_1 : X \times Y \to X \) and \( \pi_2 : X \times Y \to Y \); defined \( \pi_1(x, y) = x \) and \( \pi_2(x, y) = y \) respectively.
a) show that given a third set $Z$ and functions $f : Z \to X$, $g : Z \to Y$, there exists a function $h : Z \to X \times Y$ satisfying $h; \pi_1 = f$ and $h; \pi_2 = g$; (hint: start by drawing a diagram with all the functions)

**Solution.**

We have to find the function $h$ indicated by the dotted arrow in the diagram above that satisfies the required property. Let $h(z) \overset{\text{def}}{=} (f(z), g(z))$. Then for all $z \in Z$ we have

$$(h; \pi_1)(z) = \pi_1(h(z)) = \pi_1(f(z), g(z)) = f(z)$$

so $h; \pi_1 = f$. Similarly $(h; \pi_2)(z) = \pi_2(f(z), g(z)) = g(z)$ so $h; \pi_2 = g$.

b) show that the function $h$ is the unique such function; i.e. if there exists $h'$ with $h'; \pi_1 = f$ and $h'; \pi_2 = g$ then $h = h'$.

**Solution.** Suppose that there exists another function $h'$ that satisfies the required properties. Then $\pi_1(h'(z)) = (h'; \pi_1)(z) = f(z)$ so $h'(z)$ must be of the form $(f(z), y)$. Similarly $\pi_2(h'(z)) = g(z)$ so $h'(z)$ must be $(x, g(z))$. The only choice is $h'(z) = (f(z), g(z))$ which is also, by definition, the value of $h(z)$.

6. Suppose that $X$ and $Y$ are sets and $X + Y$ is their sum. There are two functions known as the injections: $i_1 : X \to X + Y$ and $i_2 : Y \to X + Y$; defined $i_1(x) = (x, 0)$ and $i_2(y) = (y, 1)$ respectively.

a) show that given a third set $Z$ and functions $f : X \to Z$, $g : Y \to Z$, there exists a function $h : X + Y \to Z$ satisfying $i_1; h = f$ and $i_2; h = g$;

b) show that $h$ is the unique such function.

**Solution.** Similar in spirit to the previous exercise. It is left for you to work out the details.