1. Let \( \mathbb{2} = \{0, 1\} \). List all relations on \( \mathbb{2} \) that are not functions \( \mathbb{2} \to \mathbb{2} \).

**Solution.** A relation on \( \mathbb{2} \) is a subset of the cartesian product \( \mathbb{2} \times \mathbb{2} \). There are \( 2^{2 \times 2} = 2^4 = 16 \) such subsets.

Any function \( \mathbb{2} \to \mathbb{2} \) has to be defined on its entire domain; so any function has to contain at least two elements: \((0, x)\) and \((1, y)\) for some \(x, y \in \{0, 1\}\). This rules out the following relations:

\[ \varnothing, \{(0, 0)\}, \{(0, 1)\}, \{(1, 0)\}, \{(1, 1)\} \]

The other property a function has to satisfy is to have a unique value in the codomain for each element in the domain: so if \((x, y)\) and \((x, z)\) are in the function then it must be that \(y = z\). Here this rules out all the relations with more than two elements:

\[\{(0, 0), (0, 1), (1, 0)\}, \{(0, 0), (0, 1), (1, 1)\}, \{(0, 0), (1, 0), (1, 1)\}, \{(0, 0), (0, 1), (1, 0), (1, 1)\} \]

What about the relations with two elements? There are six. (why?) The following fail to be functions, both because they do not account for the whole domain and because they are not single valued:

\[\{(0, 0), (0, 1)\}, \{(1, 0), (1, 1)\}\]

That leaves the four remaining relations that are functions:

\[\{(0, 0), (1, 0)\}, \{(0, 0), (1, 1)\}, \{(0, 1), (1, 0)\}, \{(0, 1), (1, 1)\}\]

2. What are the elements of \( \mathcal{P}(\mathcal{P}(X)) \)? Give an example of a function \( \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X) \) and \( X \to \mathcal{P}(X) \).

**Solution.** An element of \( \mathcal{P}(\mathcal{P}(X)) \) is a set of subsets of \( X \). For example, if \( X = \{0, 1\} \) then an example of an element of \( \mathcal{P}(\mathcal{P}(X)) \) is \( \varnothing, \{0\}, \{1\} \). There are several different functions \( \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X) \);
one important function is the function \( \cup : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X) \) that sends a set of subsets to their union. So, for example
\[
\cup(\{\emptyset, \{0\}, \{1\}\}) = \emptyset \cup \{0\} \cup \{1\} = \{0, 1\}.
\]

An obvious function \( X \to \mathcal{P}(X) \) is the so-called singleton function that takes \( x \) to the 1-element subset \( \{x\} \).

3. Does it hold for all finite sets \( X, Y \) that \( |\mathcal{P}(X \times Y)| = |\mathcal{P}(X)| \cdot |\mathcal{P}(Y)| \)?

Solution. It does not hold. There are many possible counterexamples. One is to take \( X \) and \( Y \) to be two singletons (sets with one element). Then the left-hand side is \( 2^1 = 2 \) and the right-hand side is \( 2^1 \cdot 2^1 = 4 \).

4. The function \( \Delta_X : X \to X \times X \) defined \( \Delta_X(x) = (x, x) \) is sometimes called the diagonal. It is always injective? When is it surjective?

Solution. It is easy to prove that \( \Delta_X \) is always injective. Indeed, suppose that for some \( x, y \in X \) it happens that \( \Delta_X(x) = \Delta_X(y) \). By definition of \( \Delta \) this means that \( (x, x) = (y, y) \). By definition of equality between ordered pairs this amounts to \( x = y \), which finishes the proof.

In general \( \Delta_X \) is not surjective. To see this, consider \( \Delta_2 : 2 \to 2 \times 2 \). Then elements \( (0, 1) \) and \( (1, 0) \) are not in the image of \( \Delta_2 \)'s.

\( \Delta \) is surjective when \( X \) is either \( \emptyset \) or a singleton.

5. Show that for any sets \( X, Y, Z \), the canonical function \( \varphi : (X \times Y) \times Z \to X \times (Y \times Z) \) \((\varphi((x, y), z) = (x, (y, z)))\) is a bijection.

Solution. We can do this by showing that \( \varphi \) is injective and surjective. Another way is to show that \( \varphi \) has an inverse, indeed if we define a function \( \psi : X \times (Y \times Z) \to (X \times Y) \times Z \) by setting \( \psi(x, (y, z)) = ((x, y), z) \) then, for all \( x \in X \), \( y \in Y \) and \( z \in Z \) we have \((\varphi; \psi)((x, y), z) = \psi(x, (y, z)) = (x, (y, z)) \) and similarly \((\psi; \varphi)(x, (y, z)) = \varphi\psi(x, (y, z)) = \varphi((x, y), z) = (x, (y, z)) \). By the isomorphism theorem, \( \varphi \) is bijective.

6. Let \( X \neq \emptyset \) and \( Y \) be sets and \( f : X \to Y \) an injective function. Define a surjective function \( g : Y \to X \) such that \( f; g = \text{id}_X \). (The converse, namely the ability, given a surjective function \( f : X \to Y \), to find an injective function \( g : Y \to X \) such that \( g; f = \text{id}_Y \), is known as the
axiom of choice).

**Solution.** If \( X \neq \emptyset \) then there exists an element \( x_0 \in X \). Define the function \( g : Y \to X \) as follows:

\[
g(y) = \begin{cases} 
  x & \text{if } y \text{ is in the range of } f \text{ and } f(x) = y; \\
  x_0 & \text{otherwise.}
\end{cases}
\]

Note that this definition makes sense only because if \( y \) is in the range of \( f \) there is precisely one \( x \in X \) such that \( f(x) = y \) (injectivity!).

To show that \( g \) is surjective we need to find, for each \( x \in X \), a \( y \in Y \) such that \( g(y) = x \). Letting \( y = f(x) \) does the job.

7. Suppose that \( X, Y \) and \( Z \) are sets. Find a bijection between \( X^Y \times X^Z \) and \( X^{Y+Z} \). You may want use the fact that there are two canonical functions \( i_1 : Y \to Y + Z \) \((i_1(y) = (y,0))\) and \( i_2 : Z \to Y + Z \) \((i_2(z) = (z,1))\).

**Solution.** We can define a function

\[
\varphi : X^Y \times X^Z \to X^{Y+Z}
\]

that takes an element of \( X^Y \times X^Z \), that is, a pair of functions \((f, g)\) where \( f : Y \to X \) and \( g : Z \to X \) to the function \( h : Y + Z \to X \) defined \( h(y,0) = f(y) \) and \( h(z,1) = g(z) \). Notice that this works as a definition of \( h \) since every element of the domain \( Y + Z \) is either of the form \((y,0)\) or \((z,1)\).

We can argue directly that \( \varphi \) is a bijection by showing it is injective and surjective, or we can define an inverse. Here we can use the injections \( i_1 : Y \to Y + Z \) and \( i_2 : Z \to Y + Z \) to get what we need. Define

\[
\psi : X^{Y+Z} \to X^Y \times X^Z
\]

by \( \psi(h) = (i_1; h, i_2; h) \). It is now straightforward to show that \( \psi \) is the inverse of \( \varphi \), that is

\[
\varphi; \psi = \text{id}_{X^Y \times X^Z}
\]

and

\[
\psi; \varphi = \text{id}_{X^{Y+Z}}
\]
8. Let $2 = \{0, 1\}$. Use diagonalisation to show that the function space $2^\mathbb{N}$ is uncountable. What does this say about $\mathcal{P}(\mathbb{N})$?

**Solution.** Recall that $2^\mathbb{N} \overset{\text{def}}{=} \{ f \mid f : \mathbb{N} \to 2 \}$: the set of all functions from $\mathbb{N} \to 2$. To show that $2^\mathbb{N}$ is not countable we show that there is no surjective function $\mathbb{N} \to 2^\mathbb{N}$.

Start with any function $f : \mathbb{N} \to 2^\mathbb{N}$. Then $f(0), f(1), f(2)$ etc are all functions $\mathbb{N} \to 2$. We can use these functions to fill in a table as follows:

\[
\begin{array}{c|cccccc}
0 & 1 & 2 & 3 & \ldots \\
0 & 0 & 1 & 0 & 0 & \ldots \\
1 & 1 & 1 & 0 & 1 & \ldots \\
2 & 0 & 0 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{array}
\]

where the $i$th row corresponds to $f(i) : \mathbb{N} \to 2$ and the value (either 0 or 1) in the $j$th column is the $f(i)(j)$. Note that I filled in the table values arbitrarily; the point is that once I have an $f$ then I could construct any finite piece of such an infinite table.

The point of drawing the table is to give an idea of how to define a function $g : \mathbb{N} \to 2$ that is not in the range of $f$; this function will be defined by “flipping” the diagonal. First we must make precise what we mean by flipping: define a function $\neg : 2 \to 2$, setting $\neg 0 \overset{\text{def}}{=} 1$ and $\neg 1 \overset{\text{def}}{=} 0$. Now we define $g$:

$$g(n) = \neg(f(n)(n)).$$

So $g$ at $n$ is the flipped value of the $n$th position in the diagonal. Then it follows that $g$ could not possibly be in the table because it is different from every function in the table in at least one spot. Indeed, for any $k \in \mathbb{N}$, $g$ cannot be $f(k)$ because $g(k) = \neg(f(k)(k)) \neq f(k)(k)$; $g$ and $f(k)$ differ at $k$. In other words, $g$ is not in the range of $f$ meaning that $f$ is not surjective.

We know that there is a bijection $\mathcal{P}(\mathbb{N}) \to 2^\mathbb{N}$ (write out the details as an exercise!). So we can conclude that the powerset of the naturals is not countable.

9. Define a relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ as follows

$$R = \{(p, q) \mid p, q \in \mathbb{Z}, p = q \text{ or } p + q = 0\}$$
(a) Show that $R$ is an equivalence relation.

**Solution.** We need to show that $R$ is reflexive, symmetric and transitive.

For reflexivity, we need to show that—for each integer $s \in \mathbb{Z}$—we have $(s, s) \in R$. This is obviously true, since $s = s$, so the predicate that defines $R$ is satisfied.

For symmetry, we need to show that if $(r, s) \in R$ then $(s, r) \in R$. If $r = s$ then there is nothing to do. If $r \neq s$ then since $(r, s) \in R$ it must be the case that $r + s = 0$. But then of course $s + r = 0$, so $(s, r) \in R$.

For transitivity, we need to show that if $(r, s) \in R$ and $(s, t) \in R$ then $(r, t) \in R$. Again, if $r = s$ or $s = t$ then there is nothing to do. The only interesting case, therefore, is when $r \neq s$ and $s \neq t$. In that case, since $(r, s) \in R$ and $(s, t) \in R$, we have that $r + s = 0$ and $s + t = 0$. The second equation is the same as saying that $s = -t$. Substituting for $s$ in the first gives $r + (-t) = 0$, so $r = t$. Therefore $(r, t) \in R$.

(b) What are the equivalence classes? Can you find a simple way of describing $\mathbb{Z}/R$?

**Solution.** The equivalence class of 0 ([0]) contains all integers that are either equal to 0 or $-0$. But of course $-0 = 0$ so $[0] = \{0\}$. For any other integer $p$ we have $[p] = \{p, -p\}$. So we have

$$\mathbb{Z}/R = \{[0], [1], [2], \ldots \}$$

that is, we have an equivalence class for each natural number.