COMP2210: Theory of Computing

Lecture 14

Decidability and Universal Turing Machines

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Objectives for Today

- Recursively Enumerable and Recursive Sets
- Decidable and Semidecidable Properties
Recall: TMs, informally

- Start in the start state $s \in Q$ at the leftmost position on the tape;
- In each step the TM is in some state $q$ and at tape position with $\gamma$. Depending on $q$ and $\gamma$ it will:
  1. write a new symbol at current tape position,
  2. move left or right and
  3. change state.
- a TM accepts by reaching a special accept state $t \in Q$, rejects by entering a special reject state $r \in Q$. 
Recursive & recursively enumerable sets

- write $L(M)$ for the set of strings accepted by $M$.
- a set of strings (a language) is called recursively enumerable when there is some TM $M$ that accepts it;
- a TM that halts on all inputs is called total;
- a set/language is called recursive when there is some total TM $M$ that accepts it.
- the set $\{ a^n b^n c^n | n \geq 0 \}$ is recursive.
  - last lecture: TM $M$ that accepts when initially the tape is of the form $\vdash a^n b^n c^n$ for any $n \in N$ and rejects otherwise.
Properties vs Languages

- Suppose $P$ is a property of strings – i.e. for any $x \in \Sigma^*$ either $P(x)$ or $\neg P(x)$

  (e.g. “has length 2”)

- $P$ is decidable $\overset{\text{def}}{=} \{ x \in \Sigma^* | P(x) \}$ is recursive.

- $P$ is semidecidable $\overset{\text{def}}{=} \{ x \in \Sigma^* | P(x) \}$ is r.e.

- notions decidable/recursive & semidecidable/r.e.
  are equivalent:

  $A$ is recursive $\iff “x \in A”$ is decidable, $A$ is r.e. $\iff “x \in A”$ is semidecidable
Recursive sets

• are accepted by *total* Turing machines;

**Lemma.** Recursive sets are closed under complement.

**Proof.** Suppose that \( A \) is accepted by \( M \). Then to accept \( \Sigma^* - A \) it suffices to construct \( M' \) that is like \( M \) but with swapped accept and reject states.
Recursively enumerable sets

- are accepted by (ordinary) Turing machines;
- *not* in general closed under complement (proof on previous slide does not work, why?)

**Lemma.** If both $A$ and $\neg A$ are r.e. then $A$ is recursive.

**Idea.** Suppose that $M$ is a TM for $A$ and $M'$ is a TM for $\neg A = \Sigma^* - A$. Construct a *total* TM $N$ that simulates running $M$ and $M'$ simultaneously.
The simulation, 1

- for all $\gamma \in \Gamma$ and $\delta \in \Gamma'$, $N$ has the following tape symbols:

- **Idea:** Each cell represents two cells – one of $M$ and one of $M'$ – “hats” mark positions of heads.

- **Simulation:** If the input is $x_1 \ldots x_k$, $N$ first rewrites the non-empty portion of tape as follows:
The simulation, 2

- \( N \) then performs the following loop:
  1. scans the tape to find a symbol with a hat in the "upper" section of the tape. Then, according to the transitions of \( M \), it performs \( M \)'s move.
  2. if \( M \) accepts then \( N \) accepts;
  3. scans the tape to find a symbol with a hat in the "lower" section of the tape. Then, according to the transitions of \( M' \), it performs \( M' \)'s move.
  4. if \( M' \) accepts then \( N \) rejects;

- Any \( x \) is either in \( A \) or \( \Sigma^* - A \). So it is accepted by either \( M \) or \( M' \). So \( N \) will eventually accept or reject! Thus \( N \) is total and so \( A \) is recursive.