Learning Outcomes

By the end of this session you should be able to...

- ... apply the pumping Lemma to show that a language is not regular
Pumping lemma

Suppose that $A$ is regular. There exists $k \in \mathbb{N}$ such that for all strings $x, y, z$ with $xyz \in A$ and $\#y \geq k$:

- there exist strings $u, v, w$ with:
  - $y = uvw$ and $v \neq \epsilon$;
  - $xuv^i wz \in A$ for all $i \geq 0$. 

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Using the pumping lemma

- we will most often use the pumping lemma to show that a language is *not* regular
- the pumping lemma is of the form:
  \[(A \text{ regular}) \Rightarrow \Phi\] (1)

- this is logically equivalent to:
  \[\neg\Phi \Rightarrow (A \text{ is not regular})\] (2)

- (2) is the *contrapositive* of (1).
Pumping lemma, contrapositive

If for all $k > 0$, there exist strings $x, y, z$ so that $xyz \in A$, $\#y \geq k$ and:

- for all strings $u, v, w$ with $y = uvw$ and $v \neq \epsilon$
  there exists $i \geq 0$ such that $xuv^i w z \notin A$

then $A$ not regular.

- the interplay between $\forall$ and $\exists$ can be understood as a game with an opponent.
The demon game

- The game proceeds as follows:
  1. demon picks some $k > 0$;
  2. we pick $x, y, z$ such that $xyz \in A$ and $\# y \geq k$;
  3. demon picks $u, v, w$ such that $y = uvw$ and $v \neq \epsilon$
  4. we pick $i \geq 0$

- we win if $xuv^i wz \not\in A$, the demon wins if $xuv^i wz \in A$.

- if we have a winning strategy (ie. no matter what the demon does, we can always win) then we have shown that $A$ is not regular.
Example 1

• Let’s play the game and show that the following is not regular:

\[ A = \{ a^n b^n \mid n \in \mathbb{N} \} \]

1. the demon picks some \( k > 0 \);
2. we pick \( x = a^k, y = b^k, z = \epsilon \);
3. the demon picks some \( u, v, w \) such that \( y = uvw \) and \( v \neq \epsilon \):
   - because of our choice of \( y \), \( v = b^l \) for some \( 0 < l \leq k \)
   - then \( u = b^r \) and \( w = b^s \) for some \( r, s \geq 0 \) with \( k = r + l + s \) (since \( y = b^k = b^r b^l b^s \)). In particular \( r + s = k - l \).
Example 1 ctd

- continuing from the previous slide:

4. we pick \( i = 0 \);
   - then \( xuv^0wz = a^kb^r b^s \epsilon = a^kb^{r+s} = a^kb^{k-l} \)

- We win because \( a^kb^{k-l} \not\in A \). This is a winning strategy, because we have made no assumption on what the demon does: whatever the demon does, we end up winning. Thus \( A \) is not regular.
Example 2 - Palindromes

- We will show that the language $P$ of palindromes is not regular. A palindrome is a string that reads equally from left to right and from right to left.

- Assume that $P$ is regular. Then also $P \cap L(a^*ba^*)$ is regular since regular languages are closed under intersection. Clearly:

\[ P \cap L(a^*ba^*) = \{ a^nba^n \mid n \in \mathbb{N} \} \]

- Using the pumping lemma, we will show that the above is not regular. This is a contradiction to our original assumption that $P$ was regular.
Palindromes II

- we will play the demon game on:

\[ A \overset{\text{def}}{=} \{ a^n ba^n \mid n \in \mathbb{N} \} \]

1. the demon picks some \( k > 0 \);
2. we pick \( x = \epsilon, y = a^k, z = ba^k \);
3. the demon divides \( y \) into \( uvw \) with \( v \) non-empty. Hence \( v = a^l \) for some \( 0 < l \leq k \);
4. we pick \( i = 0 \), clearly \( a^{k-l} ba^k \) is not in the language.

- the above is a winning strategy, and thus \( A \) is not regular.
Learning Outcomes

You should be able to . . .

• . . . *apply* the pumping Lemma to show that a language is not regular