1. Prove that the set \( \{a^n b^m c^k \mid n + m = k \} \) is not regular.

**Solution.** We will play the demon game on \( A = \{a^n b^m c^k \mid n, m, k \geq 0, n + m = k \} \).

The demon picks some \( k > 0 \). We need to pick strings \( x, y, z \) such that \( xyz \in A \) and \( \#y \geq k \). There are several choices here (we can pump the \( a \)'s, the \( b \)'s or the \( c \)'s) so let’s choose to pump the \( b \)'s: let \( x = a^k, y = b^k \) and \( z = c^{2k} \).

The demon now picks \( u, v, w \) such that \( uvw = b^k \) and \( v \neq \epsilon \). Then it must be that \( v = b^l \) for some \( l > 0 \) and \( u = b^r, w = b^s \) for some \( r, s \geq 0 \) such that \( b^r b^l b^s = b^k \), i.e. \( r + l + s = k \), i.e. \( r + s = k - l \).

We pick \( i = 0 \). Then \( xuv^0wz = a^k b^r b^s c^{2k} = a^k b^{k-l} c^{2k} \). But since \( l > 0 \), clearly \( 2k - l \neq 2k \) and so this string is not in \( A \). This is a winning strategy as we made no assumptions about what the demon does apart from requiring him to follow the rules of the game. Notice that we could have also just as well picked \( i \) to be any other natural number apart from 1.

2. For each of the following languages decide whether it is regular or not and prove your claim.

(a) \( \{a^n b^m \mid n, m > 0 \} \)

**Solution.** Regular. This is nothing but \( L(a^* b^*) \).

(b) \( \{a^n b^m \mid \exists k \in \mathbb{N}. m + n = 2^k \} \)

**Solution.** Not regular. Demon picks \( k \). We pick \( x = \epsilon, y = a^{2 \max(k, 2)}, z = \epsilon \).

Demon must pick \( v = a^p \) for some \( 0 < p \leq 2^k \). In the final step we have \( p + q = 2^k \) and must find an \( i \geq 0 \) such that \( ip + q = 2^k \). Pick \( i = q \), then \( qp + q = q(p + 1) \). Now if \( p \) was not 1 then this number has a factor which is not 2, so it cannot be \( 2^k \). If \( p \) was 1 then \( q \) was \( 2^k - 1 \) which is 1 only in the case \( k = 1 \), but we smartly avoided this in step 2 by our cunning use of \( \max \). So \( q \) is some odd number \( > 1 \), but \( 2^k \) does not have any factors different from 2.

(c) \( \{a^n b^m \mid m < n \} \)

**Solution.** Not regular. Demon picks \( k \). We pick \( x = a^{k+1}, y = b^k, z = \epsilon \).

Demon must pick \( v = b^l \) for some \( 0 < l \leq k \). We pick \( i = 2 \), then the result is \( a^{k+1} b^{k+l} \notin L \) since \( l \) is at least 1.

(d) \( \{a^n b^m \mid m < n \text{ or } n < m \} \)

**Solution.** Not regular, we know that regular languages are closed under complement and \( \overline{L} \cap L(a^* b^*) = \{a^n b^n \mid n \in \mathbb{N} \} \), which we know is not regular.