1. Consider the labelled transition system below

\[
\begin{align*}
  x_0 & \xrightarrow{c} x_3 \\
  x_1 & \xrightarrow{c} x_2 \\
  x_3 & \xrightarrow{a} x_1 \\
  x_2 & \xrightarrow{a} x_3
\end{align*}
\]

Give the set of traces from \(x_0, x_1, x_2, x_3\).

**Solution.** Traces from \(x_2\): \(\{\epsilon\}\). Traces from \(x_3\): \(\{\epsilon, a\}\). Traces from \(x_1\): \(\{\epsilon, c\}\). Traces from \(x_0\): \(\{\epsilon, a, c, ac, ca\}\).

2. Consider the labelled transition system below.

\[
\begin{align*}
  y_0 & \xrightarrow{a} y_1 \\
  y_0 & \xrightarrow{c} y_2 \\
  y_1 & \xrightarrow{a} y_2 \\
  y_2 & \xrightarrow{c} y_1
\end{align*}
\]

(a) Are \(x_0\) (from the previous question) and \(y_0\) trace equivalent?

**Solution.** Traces from \(y_0\): \(\{\epsilon, a, c, aa, ac, ca, cc\}\). Since \(\{\epsilon, a, c, ac, ca\} \neq \{\epsilon, a, c, aa, ac, ca, cc\}\), \(x_0\) and \(y_0\) are not trace equivalent.

(b) Show that \(y_0\) simulates \(x_0\).

**Solution.** We show that \(R = \{(x_0, y_0), (x_1, y_1), (x_3, y_1), (x_2, y_2)\}\) is a simulation. We need to check that each pair satisfies the defining property of simulations.

From \((x_0, y_0)\) there are two possible moves for \(x_0\). First \(x_0 \xrightarrow{a} x_1\). Here \(y_0\) can respond with \(y_0 \xrightarrow{a} y_1\) and we know that \(x_1Ry_1\). Also \(x_0 \xrightarrow{c} x_3\). Now \(y_0 \xrightarrow{c} y_1\) and \(x_3Ry_1\).

From \((x_1, y_1)\), \(x_1 \xrightarrow{c} x_2\). But \(y_1 \xrightarrow{c} y_2\) and \(x_2Ry_2\).
From \((x_2, y_2)\), there is nothing to check, since there are no moves from \(x_2\).

(c) Play the simulation game to show that \(x_0\) does not simulate \(y_0\).

**Solution.** We start in position \((y_0, x_0)\) and the demon chooses \(y_0 \xrightarrow{a} y_1\). We have to respond with \(x_0 \xrightarrow{a} x_1\). The game now continues from \((y_1, x_1)\). The demon now chooses \(y_1 \xrightarrow{a} y_2\), and we are stuck since we cannot play an \(a\) move from \(x_1\). The demon wins – note that this is a winning strategy since we had no choices to make, the demon is guaranteed to win every time, if he follows this strategy. Thus \(x_0\) does not simulate \(y_0\).

3. Show that \(x_0\) and \(y_0\) in the labelled transition system below are bisimilar.

\[
\begin{array}{cccc}
\xrightarrow{a} & \xrightarrow{a} & \xrightarrow{a} & \\
& \xrightarrow{a} & \\
\end{array}
\]

**Solution.** We show that \(R = \{ (x_0, y_0), (x_1, y_0) \}\) is a bisimulation. So for each pair, we need to verify that the condition in the definition of bisimulation is satisfies.

For \((x_0, y_0)\), \(x_0 \xrightarrow{a} x_1\), to which \(y_0\) can respond with \(y_0 \xrightarrow{a} y_0\), and \(x_1 \xrightarrow{a} x_0\). Also \(y_0 \xrightarrow{a} y_0\), to which \(x_0\) can respond with \(x_0 \xrightarrow{a} x_1\), with \(x_1 \xrightarrow{a} x_0\).

For \((x_1, y_0)\), both \(x_1\) and \(y_1\) can only loop on \(a\) back to themselves, so clearly the bisimulation condition is satisfied.

4. Consider the two labelled transition system below.

\[
\begin{array}{cccc}
x_0 \xrightarrow{c} x_3 & \xrightarrow{a} x_1 \xrightarrow{b} x_2 & \xrightarrow{c} y_0 \\
\end{array}
\]
(a) Show that \{(x_0, y_0), (x_1, y_1), (x_1, y_3), (x_3, y_3), (x_2, y_2), (x_2, y_4)\} is a simulation.

**Solution.** From \((x_0, y_0)\): If \(x_0 \xrightarrow{a} x_1\), then \(y_0 \xrightarrow{a} y_1\) and \(x_1Ry_1\). If \(x_0 \xrightarrow{c} x_3\), then \(y_0 \xrightarrow{c} y_3\) and \(x_3Ry_3\).

From \((x_1, y_1)\): If \(x_1 \xrightarrow{a} x_1\), then \(y_1 \xrightarrow{a} y_3\) and \(x_1Ry_3\). If \(x_1 \xrightarrow{b} x_2\) then \(y_1 \xrightarrow{b} y_2\) and \(x_2Ry_2\).

From \((x_1, y_3)\): If \(x_1 \xrightarrow{a} x_1\) then \(y_3 \xrightarrow{a} y_3\) and \(x_1Ry_3\). If \(x_1 \xrightarrow{b} x_2\) then \(y_3 \xrightarrow{b} y_4\) and \(x_2Ry_4\).

From \((x_3, y_3)\): If \(x_3 \xrightarrow{b} x_2\) then \(y_3 \xrightarrow{b} y_4\) and \(x_2Ry_4\).

From \((x_2, y_2)\) and \((x_2, y_4)\) there is nothing to check.

(b) Play the bisimulation game to show that \(x_0\) and \(y_0\) are not bisimilar.

**Solution.** We start in the state \((x_0, y_0)\). The demon chooses to play at \(x_0\), choosing \(x_0 \xrightarrow{a} x_3\). We have to match with \(y_0 \xrightarrow{c} y_3\).

The game continues from \((x_3, y_3)\). The demon now chooses to play at \(y_3\), and chooses \(y_3 \xrightarrow{a} y_3\). We cannot respond, since there is no \(a\) move from \(x_3\) — we lose!

5. Prove that a union of two simulations is a simulation.

**Solution.** Suppose that \(R\) and \(S\) are simulations.

We need to show that \(R \cup S\) is a simulation, i.e. that it satisfies the defining property of simulations. Suppose therefore that \((x, y) \in R \cup S\) and that \(x \xrightarrow{a} x'\). Since \((x, y) \in R \cup S\), by the definition of union, it must be the case that either \((x, y) \in R\) or \((x, y) \in S\). The argument is the same in both cases, so let’s assume (without loss of generality) that \((x, y) \in R\). Now, since \(R\) is a simulation relation, it must be the case that there exists \(y'\) and a transition \(y \xrightarrow{a} y'\), such that \((x', y') \in R\). But since \(R \subseteq R \cup S\), we also have \((x', y') \in R \cup S\). Thus \(R \cup S\) is a simulation.

6. Prove that simulation equivalence implies trace equivalences. That is, if \(x\) and \(y\) are simulation equivalent then they are capable of executing the same set of traces.

**Solution.** It’s helpful to introduce some additional notation. Given a state \(x\), let \(Tr(x)\) be the (possibly infinite) set of traces from \(x\).
To prove the claim, we will prove a slightly different result that will lead us to an easy answer.

**Lemma.** Suppose that $y$ simulates $x$. Then $Tr(x) \subseteq Tr(y)$.

To see that the above is enough, if $x$ and $y$ are simulation equivalent then it means that $y$ simulates $x$ and $x$ simulates $y$. Therefore, by the conclusion of the lemma above, $Tr(x) \subseteq Tr(y)$ and $Tr(y) \subseteq Tr(x)$. Since the two sets are included in each other, it follows that they must contain precisely the same elements, i.e. $Tr(x) = Tr(y)$.

Let us now go back and prove the lemma. Given that $y$ simulates $x$, there is a simulation relation $R$ with $xRy$. Suppose that $w \in Tr(x)$, i.e. $w$ is a trace from $x$. We will want to show that $w$ is also a trace of $y$, i.e. $w \in Tr(y)$.

We argue by induction on the length $|w|$ of $w$.

If $|w| = 0$ then it must be the case that $w = \epsilon$, the empty trace. Any two states are capable of doing the empty trace, so vacuously $w = \epsilon \in Tr(y)$.

Suppose now that the statement of the lemma is true for all traces of length $n$ and consider a trace of length $n + 1$. Therefore $w = av$ for some action $a$ and trace $v$ with $|v| = n$.

Since $w = av$ is a trace from $x$, we must have an $x'$ with

$$x \xrightarrow{a} x' \quad \text{and} \quad v \in Tr(x')$$

Given that $xRy$ and $R$ is a simulation, there exists $y'$ with $y \xrightarrow{a} y'$ and $x'Ry'$. Since $y'$ simulates $x'$, the inductive hypothesis tells us that $Tr(x') \subseteq Tr(y')$ and in particular $v \in Tr(y')$. But there is an $a$-labelled transition from $y$ to $y'$, so $w = av \in Tr(y)$. QED.