1. Linear dependence

(1) Is the vector \( \mathbf{v} \in \mathbb{R}^3 \) in the span of the set \( \{ \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \} \)?

\[
\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}?
\]

(2) The toy example in the first set of slides that mentioned recommender systems had written out the following SVD \( \mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^T \):

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & \sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & 0 \\
0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
\sqrt{\frac{1}{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0
\end{pmatrix}^T.
\]

(a) Let the columns of \( \mathbf{U} \) be denoted \( \mathbf{u}_1, \mathbf{u}_2 \) and \( \mathbf{u}_3 \), and those of \( \mathbf{V} \) denoted \( \mathbf{v}_i, \ i = 1, \ldots, 5 \). Show that these singular vectors \( \{ \mathbf{u}_i \} \) and \( \{ \mathbf{v}_i \} \) form orthonormal sets.

(b) Calculate \( \mathbf{U} \mathbf{U}^T \), \( \mathbf{U}^T \mathbf{U} \), \( \mathbf{V} \mathbf{V}^T \) and \( \mathbf{V}^T \mathbf{V} \).

(c) Express the original individual user \( \mathbf{\tilde{u}}_2 \) and movie vector \( \mathbf{\tilde{v}}_3 \) as linear combinations of these singular vectors as basis vectors. In other words, for

\[
\mathbf{\tilde{u}}_2 \triangleq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{\tilde{v}}_3 \triangleq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

Find \( \mathbf{\tilde{u}}_2 = \alpha_1^{(2)} \mathbf{u}_1 + \alpha_2^{(2)} \mathbf{u}_2 + \alpha_3^{(2)} \mathbf{u}_3 \), \( \mathbf{\tilde{v}}_3 = \beta_1^{(3)} \mathbf{v}_1 + \cdots + \beta_5^{(3)} \mathbf{v}_5 \) and \( \mathbf{\tilde{v}}_3 = \beta_1^{(3)} \mathbf{v}_1 + \cdots + \beta_5^{(3)} \mathbf{v}_5 \). (Note the difference between \( \mathbf{\tilde{u}}_i \) and \( \mathbf{u}_i \), etc.)
2. Matrix polynomials

(1) For \( A = \begin{pmatrix} -4 & 2 \\ 3 & 1 \end{pmatrix} \), and \( f(x) = x^2 + 3x - 10 \), calculate \( f(A) \).

- The answer \( f(A) \) is a matrix.
- \( 3A \) is a matrix.
- The number 10 in \( f(x) \) has to be multiplied by the \( 2 \times 2 \) identity matrix to make it a matrix.
- **Hint:** Verify \( A^2 = \begin{pmatrix} 22 & -6 \\ -9 & 7 \end{pmatrix} \).

(2) Solve for \( x \): \( f(x) = x^2 + 3x - 10 = 0 \). Call the solutions \( x_1 \) and \( x_2 \).

(3) Define the matrices \( B_1 = A - x_1I \) and \( B_2 = A - x_2I \) where \( I \) is the \( 2 \times 2 \) identity matrix. Evaluate the determinants of \( B_1 \) and \( B_2 \) They should both be zero.

(4) The columns of \( B_1 \) and \( B_2 \) must thus be linearly dependent. Find numbers \( v_1 \) and \( v_2 \) such that

\[
v_1 \times (B_1)_{col\ 1} + v_2 \times (B_1)_{col\ 2} = 0.
\]

Similarly, find numbers \( w_1 \) and \( w_2 \) such that

\[
w_1 \times (B_2)_{col\ 1} + w_2 \times (B_2)_{col\ 2} = 0.
\]

(5) **Partial answer:** \( v_1 = -2, v_2 = 1 \).

3. Computing eigenvalues and eigenvectors

The eigenvalue problem \( Ax = \lambda x \) is the following: find, for a matrix \( A \), the eigenvectors \( x \) and eigenvalues \( \lambda \).

(1) Find the eigenvalues and eigenvectors of

\[
A = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & 2 & -4 \end{pmatrix}
\]

- **STEP I:** Compute the characteristic polynomial of \( A \) and find its roots. Verify:

\[
\chi_A(\lambda) = \det(A - \lambda I) = -\lambda^3 - \lambda^2 + 10\lambda + 10
\]

and note that \( \chi_A(\lambda) = (\lambda^2 - 10)(\lambda + 1) \).

What are the eigenvalues of \( A \)?

- **STEP II:**
For each eigenvalue $\lambda_i, i = 1, 2, 3$, we need to compute the corresponding eigenvectors. Find $x, y, z$ so that

$$
\begin{pmatrix}
2 & -2 & 3 \\
0 & 1 & -3 \\
2 & 2 & -4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \lambda_i
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}.
$$

- Hint: The 3 eigenvectors are:

$$\frac{1}{(1 + \sqrt{10})}
\begin{pmatrix}
\frac{3}{2}(\sqrt{10} - 4) \\
3 \\
1 + \sqrt{10}
\end{pmatrix}, \quad 
\frac{1}{(\sqrt{10} - 1)}
\begin{pmatrix}
\frac{3}{2}(4 + \sqrt{10}) \\
-3 \\
\sqrt{10} - 1
\end{pmatrix}, \quad (0, 3)
$$

4. Singular Value Decomposition

For a matrix $X$:

$$X = \begin{pmatrix}
-1 & 2 & -1 & -3 \\
2 & 1 & 3 & 1
\end{pmatrix},$$

the singular value decomposition (SVD) of $X$ is written as $X = U\Sigma V^T$ where

$$U = \begin{pmatrix}
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}, \quad \Sigma = \begin{pmatrix}
\sqrt{21} & 0 & 0 & 0 \\
0 & 3 & 0 & 0
\end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix}
\sqrt{\frac{3}{14}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{5}{3\sqrt{7}} \\
-\frac{1}{\sqrt{12}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{7}} \\
2\sqrt{\frac{2}{21}} & \frac{\sqrt{2}}{3} & 0 & \frac{5}{3\sqrt{7}} \\
2\sqrt{\frac{2}{21}} & -\frac{\sqrt{2}}{3} & \frac{1}{\sqrt{3}} & -\frac{2}{3\sqrt{7}}
\end{pmatrix}$$

(1) Calculate $C = XX^T$. You will find that $C = \begin{pmatrix}
15 & -6 \\
-6 & 15
\end{pmatrix}$. The negative off-diagonal elements of $C$ capture the observation that for most cases, the elements of each column of $X$ are of opposite sign.

(2) Compute the eigenvalues of $C$. Solve for the equation that sets the characteristic polynomial of $C$ to zero. In other words,

- calculate $\chi_C(x) := \det(C - xI)$ and find the values $x = x_1, x_2$ such that $\chi_C(x_1) = \chi_C(x_2) = 0$.
- Hint: You will find that $\chi_C(x) = x^2 - 30x + 189$, and you should use the observation that $189 = 21 \times 9$.

(3) How do the eigenvalues $x_{1,2}$ relate to the diagonal entries of $\Sigma$?

(4) Verify that the matrices $D_i = C - x_iI$ are proportional to

$$\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},$$

and find the nullspace for each, i.e., find $v_1$ and $v_2$ such that $D_i v_i = 0$. These $v_i$s are the eigenvectors of $C$. Normalise them and compare with $U$. 


(5) You might want the help of some software for this, e.g., numpy.linalg.eig. You can check that
\[
X^TX = \begin{pmatrix}
5 & 0 & 7 & 5 \\
0 & 5 & 1 & -5 \\
7 & 1 & 10 & 6 \\
5 & -5 & 6 & 10
\end{pmatrix},
\]
which should have 4 eigenvalues. Two of them should be the same as those of \( C \). What about the other two? Verify that the un-normalised eigenvectors of \( X^TX \) are
\[
(3, -1, 4, 4)^T, (-1, -3, -2, 2)^T, (-1, 1, 0, 1)^T, (-7, -1, 5, 0)^T,
\]
and that normalising them will yield the columns of \( V \).

5. LOW-RANK APPROXIMATION

We can construct the rank-1 approximation \( \tilde{X}_1 \) of \( X \) by setting \( \tilde{X}_1 = u_1 \sigma_1 v_1^T \).

(1) Using (from the previous exercise)
\[
u_1 = \frac{1}{\sqrt{2}}(-1, 1)^T, \quad \sigma_1 = \sqrt{21}, \quad v_1 = \frac{1}{\sqrt{42}}(3, -1, 4, 4)^T
\]
confirm that the rank-1 approximation is
\[
\tilde{X}_1 = \begin{pmatrix}
-\frac{3}{2} & \frac{1}{2} & -2 & -2 \\
\frac{3}{2} & -\frac{1}{2} & 2 & 2
\end{pmatrix}.
\]
In particular, note that the rows are not independent.

(2) Compute the rank one approximation to \( C \) as \( \tilde{C}_1 = \tilde{X}_1 \tilde{X}_1^T \). This should be proportional to
\[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}.
\]
Given \( \tilde{X}_1 \), why is this not a surprise? What are its eigenvalues and eigenvectors?

(3) Verify that \( \tilde{X}_1^T \tilde{X}_1 \) is
\[
\begin{pmatrix}
\frac{9}{2} & -\frac{3}{2} & 6 & 6 \\
-\frac{3}{2} & \frac{1}{2} & -2 & -2 \\
6 & -2 & 8 & 8 \\
6 & -2 & 8 & 8
\end{pmatrix}.
\]
What would you expect its eigenvalues to be? Check that all the rows and columns are multiples of \((-\frac{3}{2}, \frac{1}{2}, -2, -2)\). Relate this observation to the eigenvalue spectrum and the definition of rank.