COMP6212: Computational Finance

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This module is in two parts:

- **Part I** Financial data analysis, taught by Prof. M. Niranjan
- **Part II** Crypto-currencies and blockchain technology, taught by Dr Jie Zhang

Spring Semester 2017/2018
Financial Equilibrium
Caution: A peculiar and rather personal view

Generate products and services

In need of

- stability against fluctuations (e.g. demand, exchange rate)
- capital investment (e.g. to modernise, grow)

Process wealth & capital

Driven by gambling instinct and greed
Finance gets bad publicity; bankers and fund managers are sometimes disliked.
The Setting

- Finance gets bad publicity; bankers and fund managers are sometimes disliked
- The system can fail badly
- When the system fails, large amounts of tax-payer money is used to bail them out. I don’t like this!
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• Yet the system is useful
  • Investors interested in future returns
    • Greed?
    • Pay for retirement
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What are the sources of computational problems?
- Time - present value of money.
- Uncertainty - of the future.
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Overview of the Module

Topics in Part I: Financial Data Analysis

- Portfolio Optimization
- Derivatives Pricing

Keywords:
- Mean-Variance optimization
- Linear and quadratic programming
- Multivariate Gaussian distribution
- Constrained optimization
- Value at risk and Conditional value at risk
- Sharpe ratio
- Present value
- Stochastic differential equations
- Ito's Lemma
- Black-Scholes model
- Options pricing
- Stochastic Simulations and Monte Carlo methods.
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Resources

Numerical Methods in Finance and Economics
A MATLAB-Based Introduction
Second Edition

Options, Futures, and Other Derivatives
Sixth Edition
John C. Hull
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Sixth Edition
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Paul Wilmott Introduces Quantitative Finance
Second Edition
Resources

- Options, Futures, and Other Derivatives, Sixth Edition
- Paul Wilmott Introduces Quantitative Finance, Second Edition
- Time Series Analysis

- plus several academic papers.
Financial Instruments (broad classes)

Bonds
- Debt instrument to raise capital;
- Delivers periodic payment (coupon);
- Has a face value on maturity;
- No ownership associated.

Stocks
- Own a small share of a company;
- Ownership may be traded in the market;
- Owning the share might earn dividends.

Derivatives
- Contracts written on the basis of a future value of a stock, currency etc.;
- Usually there is a time of maturity and a promised payoff in the contract;
- Variations in style of exercising the contract.
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Time: Present Value

- Wealth \( W_0 \) deposit in bank and get \( W_1 \) after one year
- \( W_1 = (1 + r) W_0 \), \( r \) interest rate
- Compound interest over \( n \) years: \( W_n = (1 + r)^n W_0 \)
Wealth $W_0$ deposit in bank and get $W_1$ after one year

$W_1 = (1 + r) W_0$, \hspace{0.5cm} r \text{ interest rate}$

Compound interest over $n$ years: 

$W_n = (1 + r)^n W_0$

Define interest rate as $r$ per year; allow compounding at $m$ intervals within the year

$W_1 = \left(1 + \frac{r}{m}\right)^m W_0$
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Continuous compounding $m \to \infty$

$$W_1 = \exp(r) W_0$$
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- Define interest rate as $r$ per year; allow compounding at $m$ intervals within the year
  \[ W_1 = \left(1 + \frac{r}{m}\right)^m W_0 \]
- Continuous compounding $m \to \infty$
  \[ W_1 = \exp(r) W_0 \]
- Present value of your promise to give me cash $C$ in time $t$ is
  \[ \exp(-rt) C \]
Portfolios:

- Notion of expected return and risk in investing - balancing it out

- Investing in a portfolio of assets than in a single asset - "not all eggs in one basket"

- Optimization techniques we will learn and use
  - Linear programming
  - Quadratic programming
  - (Second order cone programming)
  - Inducing sparsity – $l_1$ or lasso regularization
  - Convex optimization using CVX toolbox

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Various Topics We Will Learn
Part I (Topic I): Portfolio Optimization

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- Brownian motion, Geometric Brownian motion
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\[
\frac{dS}{S} = \mu \, dt + \sigma \, dZ
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dZ = \phi \sqrt{dt}, \quad \phi \sim (0, 1)
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Various Topics We Will Learn (cont’d)
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- Ito’s Lemma: Function of a Geometric Brownian Motion

\[
dG = \left( \mu S \frac{\partial G}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t} \right) \, dt + \sigma S \frac{\partial G}{\partial S} \, dZ
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- Black-Scholes: options pricing under specific assumptions
- Monte Carlo / Stochastic simulations: general cases
- $r_i(t)$ Return on asset $i$ at time $t$; i.e. invest at time $t-1$, what have you earned at time $t$?
Part I: Portfolio Optimization

- \( r_i(t) \) Return on asset \( i \) at time \( t \); i.e. invest at time \( t-1 \), what have you earned at time \( t \)?
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- The mean is what we expect (on average) to gain by investing.
- We think of variance in return as risk.
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- When we look at more than one asset, we can think of how returns on them are correlated: $\sigma_{ij}^2$. 

\[ \text{Return on the portfolio: } r_p = \sum_{i=1}^{N} \pi_i r_i = \pi^t \text{r} \]

Returns on the portfolio has a multivariate Gaussian distribution

\[ r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN} \end{bmatrix} \]
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- A portfolio (investment in $N$ assets) with relative weights $\pi_i$. 

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$$
\begin{pmatrix}
    r_1 \\
    r_2 \\
    \vdots \\
    r_N
\end{pmatrix}
= \begin{pmatrix}
    \mu_1 \\
    \mu_2 \\
    \vdots \\
    \mu_N
\end{pmatrix} + 
\begin{pmatrix}
    \sigma_{11} \\
    \sigma_{12} \\
    \vdots \\
    \sigma_{1N}
\end{pmatrix} + 
\begin{pmatrix}
    \sigma_{21} \\
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\cdots + 
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- Returns on the portfolio has a multivariate Gaussian distribution

$$r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}$$
Gaussian Density

\[ p(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{(x - m)^2}{2\sigma^2} \right\} \]
The Gaussian density function is given by:

\[
p(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{(x - m)^2}{2\sigma^2} \right\}
\]

Here is the MATLAB code to plot the Gaussian density function:

```matlab
x = linspace(-5,5,50);
m = 0;
s = 1;
y = normpdf(x,m,s);
figure(1), clf
plot(x,y,'LineWidth',3);
grid on
```
Multivariate Gaussian Distribution

\[ p(x) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\} \]

Parameters: mean vector \( \mu \), covariance matrix \( \Sigma \)

How do these shapes change with \( \mu \) and \( \Sigma \)?
Multivariate Gaussian Distribution

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Linear transform of multivariate Gaussian

\[ x \sim \mathcal{N}(\mu, \Sigma); \quad y = Ax \quad \implies \quad y \sim \mathcal{N}(A\mu, A\Sigma A^t) \]
Portfolio Return

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- Return on our portfolio is a linear transform of the vector of returns
  \[ r_P = \pi^T r \]
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- We can immediately write down the distribution of the return on the portfolio

\[ r_P \sim \mathcal{N}\left(\pi^T \mu, \pi^T \Sigma \pi\right) \]
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- Mean return \( M = \pi^T \mu \) and the variance on it \( V = \pi^T \Sigma \pi \)
Portfolio Return

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  \[ x \sim \mathcal{N}(\mu, \Sigma); \quad y = Ax \quad \Rightarrow \quad y \sim \mathcal{N}(A\mu, A\Sigma A^t) \]

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- Mean return \( M = \pi^T \mu \) and the variance on it \( V = \pi^T \Sigma \pi \)
- When \( \pi \) changes, \( M \) and \( V \) change — how?
As we change $\pi$ (i.e. invest in different proportions), $M$ and $V$ change.
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Not all $M$ and $V$ are realizable.

We can formulate constrained optimization problems:

For a given risk we tolerate, what is the highest return we can expect:

$$\max \pi \pi^T \mu$$

subject to

$$\pi^T \Sigma \pi = \sigma^2$$

If we hope for (expect) a given return, at what minimum risk can we achieve it?

$$\min \pi \pi^T \Sigma$$

subject to

$$\pi^T \mu = r_0$$

Other constraints possible:

$$\sum_i \pi_i = 1, \pi_i \geq 0, \alpha \leq \pi_i \leq \beta$$
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Efficient Portfolio

- As we change \( \pi \) (i.e. invest in different proportions), \( M \) and \( V \) change
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- Other constraints possible: \( \sum_{i=1}^{N} \pi_i = 1, \pi_i \geq 0, \alpha \leq \pi_i \leq \beta \)
We estimate $\mu$ and $\Sigma$ from historic data and apply optimization to allocate assets.

We hope the past might be a good reflection of future!
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Estimation:

$$\hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} r(t)$$

$$\hat{\Sigma} = \frac{1}{T} \sum_{t=1}^{T} (r(t) - \hat{\mu}) (r(t) - \hat{\mu})^T$$
Solving quadratic program in MATLAB

\[ \min_x x^T H x + f^T x \quad \text{such that} \quad \begin{cases} A x \leq 0 \\ A_{eq} x = b_{eq} \\ lb \leq x \leq ub. \end{cases} \]

\[ x = \text{quadprog}(H, f, A, b, A_{eq}, b_{eq}, lb, ub, x0) \]
Solving quadratic program in MATLAB

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Do \texttt{doc quadprog} in MATLAB and read more.
Solving quadratic program in MATLAB

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\[ x = \text{quadprog}(H, f, A, b, A_{eq}, b_{eq}, lb, ub, x0) \]

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For the portfolio optimization problem, we might have:

\[ \pi = \text{quadprog}(\Sigma, [], [], [], \mu', r_{Max}, 0, 1, []) \]
Solving quadratic program in MATLAB

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Map the problem variables to the function in the tool.
Given $\mu$ and $\Sigma$

What portfolio has the highest return, unconstrained by risk?

$$\max \pi^T \mu \quad \text{subject to} \quad \sum_{i=1}^{N} \pi_i = 1, \text{ and } \pi_i \geq 0$$

Linear Programming:

$$\min x^T f \quad \text{such that} \quad \begin{cases} A x \leq b \\ A_{eq} x = b_{eq} \\ lb \leq x \leq ub \end{cases}$$

$$w_1 = \text{linprog}(-\mu, [], [], \text{ones}(1,N), 1, 0, 0);$$

$$r_1 = w_1 \times \mu;$$
Efficient Frontier

- Given $\mu$ and $\Sigma$
- What portfolio has the highest return, unconstrained by risk?

$$\max \pi^T \mu \quad \text{subject to} \quad \sum_{i=1}^{N} \pi_i = 1, \quad \text{and} \quad \pi_i \geq 0$$

Linear Programming:

$$\min f^T x \quad \text{such that} \begin{cases} A x \leq b \\ A_{eq} x = b_{eq} \\ lb \leq x \leq ub \end{cases}$$

$$x = \text{linprog}( f, A, b, A_{eq}, b_{eq}, lb, ub )$$

$$w1 = \text{linprog}( -\mu, [], [], \text{ones}(1,N), 1, 0, 0 );$$

$$r1 = w1 * \mu;$$
What portfolio has lowest variance (unconstrained by expectation)?

\[ \text{min } \pi^T \Sigma \pi \quad \text{subject to } \sum_{i=1}^{N} \pi_i = 1 \]
What portfolio has lowest variance (unconstrained by expectation)?

\[
\min \pi^T \Sigma \pi \quad \text{subject to} \quad \sum_{i=1}^{N} \pi_i = 1
\]

```matlab
w2 = quadprog(mu, zeros(N, 1), [], [], ones(1, N), 1, zeros(N, 1), [], []);
r2 = w2' * mu;
```

- Portfolios on the efficient frontier will have returns in range \( r_1 \) to \( r_2 \)
What portfolio has lowest variance (unconstrained by expectation)?

\[
\min \pi^T \Sigma \pi \quad \text{subject to} \quad \sum_{i=1}^{N} \pi_i = 1
\]

\[
w_2 = \text{quadprog}(\mu, \text{zeros}(N, 1), [], [], \text{ones}(1, N), 1, \text{zeros}(N, 1), [], []);
\]

\[
r_2 = w_2' \ast \mu;
\]

- Portfolios on the efficient frontier will have returns in range \( r_1 \) to \( r_2 \)
- Now, select a number of points (returns we need), and find the minimum variance portfolios that will offer these returns!
What portfolio has lowest variance (unconstrained by expectation)?

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\]

- Portfolios on the efficient frontier will have returns in range \(r_1\) to \(r_2\)
- Now, select a number of points (returns we need), and find the minimum variance portfolios that will offer these returns!

\[
M = \text{linspace}(r_1, r_2, p)
\]

\[
\text{for } j=1:p
\]

\[
\text{ret} = M(j);
\]

\[
w = \text{quadprog}(\ldots);
\]

\[
V(j) = w' \ast \Sigma \ast w;
\]

end
function [PRisk, PRoR, PWts] = NaiveMV(ERet, ECov, NPts)
ERet = ERet(:); % makes sure it is a column vector
NAssets = length(ERet); % get number of assets
% vector of lower bounds on weights
V0 = zeros(NAssets, 1);
% row vector of ones
V1 = ones(1, NAssets);
% set medium scale option
options = optimset('LargeScale', 'off');
% Find the maximum expected return
MaxReturnWeights = linprog(-ERet, [], [], V1, 1, V0);
MaxReturn = MaxReturnWeights' * ERet;
% Find the minimum variance return
MinVarWeights = quadprog(ECov, V0, [], [], V1, 1, V0, [], [], options);
MinVarReturn = MinVarWeights' * ERet;
MinVarStd = sqrt(MinVarWeights' * ECov * MinVarWeights);
% check if there is only one efficient portfolio
if MaxReturn > MinVarReturn
    RTarget = linspace(MinVarReturn, MaxReturn, NPts);
    NumFrontPoints = NPts;
else
    RTarget = MaxReturn;
    NumFrontPoints = 1;
end
...%
Store first portfolio
PRoR = zeros(NumFrontPoints, 1);
PRisk = zeros(NumFrontPoints, 1);
PWts = zeros(NumFrontPoints, NAssets);
PRoR(1) = MinVarReturn;
PRisk(1) = MinVarStd;
PWts(1,:) = MinVarWeights(:)';
% trace frontier by changing target return
VConstr = ERet';
A = [V1 ; VConstr ];
B = [1 ; 0];
for point = 2:NumFrontPoints
B(2) = RTarget(point);
Weights = quadprog(ECov,V0,[],[],A,B,V0,[],[],options);
PRoR(point) = dot(Weights, ERet);
PRisk(point) = sqrt(Weights'*ECov*Weights);
PWts(point, :) = Weights(:)';
end
Summary: what have we achieved?

Mean Variance Portfolio

Portfolio Return vs. Portfolio Risk graph.
Three assets with the following properties:

\[
m = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.15 \end{bmatrix}; \quad C = 100 \times \begin{bmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{bmatrix};
\]

Study the code of function \texttt{NaiveMV} and draw the efficient frontier.

Use the function \texttt{frontcon} in MATLAB and draw the efficient frontier.
Estimation of Parameters

Estimate parameters $\mu$ and $C$ from data within a window

Optimize portfolio, invest and wait
Periodically re-balance the portfolio (re-estimate parameters)

Trade-off:
Need long window for accurate estimation
But relationships may not be stationary over long durations

Shrinkage in covariance estimates

Analysis window
to estimate mean and covariance

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- Need long window for accurate estimation
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Advances on the Mean-Variance Portfolio

Do such portfolios make money?


Including transaction costs into the optimization


Forcing the portfolio to be sparse and stable


Optimizing the execution of trade

TBC

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COMP6212
Advances on the Mean-Variance Portfolio

Do such portfolios make money?

Advances on the Mean-Variance Portfolio

- Do such portfolios make money?


- Including transaction costs into the optimization

Do such portfolios make money?


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Advances on the Mean-Variance Portfolio

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- Optimizing the execution of trade
  
  TBC
Portfolio Performance

- Sharpe Ratio: mean to standard deviation of portfolio return

\[ S = \frac{m - r}{\sigma} \]

- \( r \) “risk free” interest rate
Portfolio Performance

- Sharpe Ratio: mean to standard deviation of portfolio return

\[ S = \frac{m - r}{\sigma} \]

\( r \) “risk free” interest rate

- Value at Risk (VAR): Value such that probability of loss exceeding this is 0.01.

\[ V \text{ such that } P[-G > V] = 0.01 \]

“if a portfolio of stocks has a one-day 5% VaR of 1 million, there is a 0.05 probability that the portfolio will fall in value by more than 1 million over a one day period”
Portfolio Performance

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"if a portfolio of stocks has a one-day 5% VaR of 1 million, there is a 0.05 probability that the portfolio will fall in value by more than 1 million over a one day period"

ValueAtRisk = portvrisk(PortReturn, PortRisk, RiskThreshold, PortValue)
Portfolio Performance

- Sharpe Ratio: mean to standard deviation of portfolio return
  \[ S = \frac{m - r}{\sigma} \]
  - \( r \) “risk free” interest rate
- Value at Risk (VAR): Value such that probability of loss exceeding this is 0.01.
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ValueAtRisk=portvrisk(PortReturn,PortRisk,RiskThreshold,PortValue )

- cVAR: Conditional Value at Risk (later)
Comparison of a number of portfolio optimization methods

\[ \frac{1}{N} \text{ with re-balancing} \]

Sample based mean-variance
Comparison of a number of portfolio optimization methods

- $\frac{1}{N}$ with re-balancing
- Sample based mean-variance
- Bayesian methods (of shrinking estimates)
Empirical Evaluation
DeMiguel, J. et al. (2009),

- Comparison of a number of portfolio optimization methods
  - $\frac{1}{N}$ with re-balancing
  - Sample based mean-variance
  - Bayesian methods (of shrinking estimates)
  - Constraints
Empirical Evaluation
DeMiguel, J. et al. (2009),

- Comparison of a number of portfolio optimization methods
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  - Combination of portfolios (model averaging / mixing)
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- Comparison of a number of portfolio optimization methods
  - $\frac{1}{N}$ with re-balancing
  - Sample based mean-variance
  - Bayesian methods (of shrinking estimates)
  - Constraints
  - Combination of portfolios (model averaging / mixing)
- No method \textit{consistently} beats the naive strategy!
Sparse Portfolios
Brodie et al. (2007) PNAS

Expected return and covariance:
\[ r_t = \begin{pmatrix} r_1, t \\ r_2, t \\ \vdots \\ r_N, t \end{pmatrix} \]
\[
E[r_t] = \mu \\
E[(r_t - \mu)(r_t - \mu)^T] = C
\]

Markowitz portfolio
\[
\min_w \quad w^T C w \\
\text{subject to} \quad w^T \mu = \rho \quad \text{and} \quad 1^T_N w = 1
\]

Short selling allowed; i.e. \( w_j \) need not be positive

Covariance:
\[ C = E[r_t r_T t - \mu \mu] \]
Sparse Portfolios
Brodie et al. (2007) PNAS

- $N$ assets; $\mathbf{r}_t$, return vector at time $t$
- Expected return and covariance:

\[
\mathbf{r}_t = \begin{pmatrix}
    r_{1,t} \\
    r_{2,t} \\
    \vdots \\
    r_{N,t}
\end{pmatrix}
\]

\[
\mathbb{E}[\mathbf{r}_t] = \boldsymbol{\mu} \quad \mathbb{E}
\left[
    (\mathbf{r}_t - \boldsymbol{\mu}) (\mathbf{r}_t - \boldsymbol{\mu})^T
\right] = \mathbf{C}
\]
\( N \) assets; \( r_t \), return vector at time \( t \)

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\[
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Markowitz portfolio

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\begin{aligned}
\min_w \quad & w^T C w \\
\text{subject to} \quad & w^T \mu = \rho \text{ and } 1_N^T w = 1
\end{aligned}
\]
$N$ assets; $\mathbf{r}_t$, return vector at time $t$

Expected return and covariance:

$$\mathbf{r}_t = \begin{pmatrix} r_{1,t} \\ r_{2,t} \\ \vdots \\ r_{N,t} \end{pmatrix}$$

$$E[\mathbf{r}_t] = \mu \quad E[(\mathbf{r}_t - \mu)(\mathbf{r}_t - \mu)^T] = \mathbf{C}$$

Markowitz portfolio

$$\begin{cases} 
\min_{\mathbf{w}} \mathbf{w}^T \mathbf{C} \mathbf{w} \\
\text{subject to } \mathbf{w}^T \mathbf{\mu} = \rho \text{ and } \mathbf{1}_N^T \mathbf{w} = 1
\end{cases}$$

Short selling allowed; i.e. $w_j$ need not be positive
Sparse Portfolios
Brodie et al. (2007) PNAS

- $N$ assets; $r_t$, return vector at time $t$
- Expected return and covariance:

$$r_t = \begin{pmatrix}
    r_{1,t} \\
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    \vdots \\
    r_{N,t}
\end{pmatrix}$$

$$E[r_t] = \mu$$

$$E[(r_t - \mu)(r_t - \mu)^T] = C$$

- Markowitz portfolio

$$\begin{cases}
    \min_w w^T C w \\
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- Covariance: $C = E[r_t r_t^T - \mu \mu^T]$
Mean and covariance estimated from data (expectations as sample averages):
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\[ \hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} r_t \]
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\[ R^{T} \times N \] matrix with rows as \( r_t^T \)
Mean and covariance estimated from data (expectations as sample averages):

\[ \hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} r_t \]

\[ R \in T \times N \] matrix with rows as \( r_t^T \)

Optimization problem rewritten as

\[
\begin{cases}
\hat{w} = \min_w \frac{1}{T} \| \rho 1_T - R w \|^2_2 \\
\text{subject to } w^T \hat{\mu} = \rho, \ w^T 1_N = 1
\end{cases}
\]
Mean and covariance estimated from data (expectations as sample averages):

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\end{align*}
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Often there is strong correlation between returns
- Assets in the same sector respond in similar ways
Mean and covariance estimated from data (expectations as sample averages):
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Often there is strong correlation between returns
- Assets in the same sector respond in similar ways
- Strong correlations make \( R \) ill-conditioned \( \Rightarrow \) numerically unstable optimization
- Solution: regularization
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Often there is strong correlation between returns
- Assets in the same sector respond in similar ways

Strong correlations make \( R \) ill-conditioned \( \Rightarrow \) numerically unstable optimization

Solution: regularization

Brodie et al. suggest \( l_1 \) regularizer

\[
\begin{align*}
\hat{w} &= \min_w \left[ \| \rho 1_T - R w \|_2^2 + \tau \| w \|_1 \right] \\
\text{subject to } w^T \hat{\mu} &= \rho, \ w^T 1_N = 1
\end{align*}
\]
Passive investor, wishing to get the same return as stock index (e.g. FTSE100)
Invest in all 100 stocks of the FTSE?
Transaction costs very high
Can we find a small subset of the 100 stocks (say 10), that will approximate the performance of the index?
Subset selection / cardinality constrained optimization
\[
\begin{align*}
\min & \quad \|y - Rw\|^2_2 \\
\text{subject to} & \quad \|w\|_0 = w_0
\end{align*}
\]

The above is of combinatorial complexity
Suboptimal algorithm: greedy search
Passive investor, wishing to get the same return as stock index (e.g. FTSE100)
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0\textsuperscript{th} norm \rightarrow number of nonzero elements of \( w \) \rightarrow subset of assets
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\min_w \left[ \| y - R w \|_2^2 \right] \\
\text{subject to } \| w \|_0 = w_0
\end{aligned}
\]

\(0^\text{th}\) norm \(\rightarrow\) number of nonzero elements of \(w\) \(\rightarrow\) subset of assets

The above is of combinatorial complexity
Passive investor, wishing to get the same return as stock index (e.g. FTSE100)

- Invest in all 100 stocks of the FTSE?
- Transaction costs very high
- Can we find a small subset of the 100 stocks (say 10), that will approximate the performance of the index?
- Subset selection / cardinality constrained optimization

$$\begin{align*}
\min_w \left[ \| y - R w \|^2_2 \right] \\
\text{subject to } \| w \|_0 = w_0
\end{align*}$$

- $0^{th}$ norm $\rightarrow$ number of nonzero elements of $w \rightarrow$ subset of assets
- The above is of combinatorial complexity
- Suboptimal algorithm: greedy search
A convenient proxy to achieve sparsity is lasso ($l_1$ constrained regression)

$$\min_w \left[ \|y - Rw\|_2^2 + \tau \|w\|_1 \right]$$

Several elements of $w$ will be zero.

Tune $\tau$ to achieve different levels of sparsity.

Can incorporate transaction costs into the optimization.

Transaction costs: Usually have fixed (overhead) part and transaction-dependent part. Institutional investors fixed part negligible. Small investors can assume fixed cost only.
A convenient proxy to achieve sparsity is lasso ($l_1$ constrained regression)

$$\min_w \left[ ||y - Rw||_2^2 + \tau ||w||_1 \right]$$

Several elements of $w$ will be zero
A convenient proxy to achieve sparsity is *lasso* ($l_1$ constrained regression)

$$
\min_w \left[ ||y - Rw||_2^2 + \tau ||w||_1 \right]
$$

- Several elements of $w$ will be zero
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A convenient proxy to achieve sparsity is \textit{lasso} ($l_1$ constrained regression)

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\min_w \left[ \| y - Rw \|^2 + \tau \| w \|_1 \right]
\]

Several elements of $w$ will be zero

Tune $\tau$ to achieve different levels of sparsity

Can incorporate transaction costs into the optimization

\[
\min_w \left[ \| y - Rw \|^2 + \tau \sum_{i=1}^{N} s_i | w_i | \right]
\]
A convenient proxy to achieve sparsity is \textit{lasso} ($l_1$ constrained regression)

$$\min_{\mathbf{w}} \left[ \| \mathbf{y} - R \mathbf{w} \|_2^2 + \tau \| \mathbf{w} \|_1 \right]$$

Several elements of $\mathbf{w}$ will be zero
Tune $\tau$ to achieve different levels of sparsity
Can incorporate transaction costs into the optimization

$$\min_{\mathbf{w}} \left[ \| \mathbf{y} - R \mathbf{w} \|_2^2 + \tau \sum_{i=1}^{N} s_i | w_i | \right]$$

Transaction costs:
- Usually have fixed (overhead) part and transaction-dependent part
- Institutional investors fixed part negligible
- Small investors can assume fixed cost only
We are holding a portfolio $w$. We want to make an adjustment $\Delta w$, new portfolio $w + \Delta w$.

Transaction costs only on the adjustments $\{\Delta w\} = \min \{\Delta w | ||\rho_1^T - R(w + \Delta w)||^2_2 + \tau ||\Delta w||_1\}$

subject to $\Delta T w \hat{\mu} = 0$ and $\Delta T w 1_N = 1$
We are holding a portfolio \( w \).
We are holding a portfolio \( w \)

We want to make an adjustment \( \Delta w \), new portfolio \( w + \Delta w \)
We are holding a portfolio $\mathbf{w}$

We want to make an adjustment $\Delta \mathbf{w}$, new portfolio $\mathbf{w} + \Delta \mathbf{w}$

Transaction costs only on the adjustments

$$
\begin{align*}
\Delta \mathbf{w} &= \min_{\Delta \mathbf{w}} \left[ \left\| \rho \mathbf{1}_T - \mathbf{R} (\mathbf{w} + \Delta \mathbf{w}) \right\|_2^2 + \tau \left\| \Delta \mathbf{w} \right\|_1 \right] \\
\text{subject to } \Delta^T \mathbf{w} \hat{\mu} &= 0 \text{ and } \Delta^T \mathbf{1}_N = 1
\end{align*}
$$
Homework:

Coursework 1 will involve confirming some claims in Brodie et al.’s paper. Please download the paper and start reading.
We will use the CVX toolbox within MATLAB to implement optimization

http://cvxr.com/
We will use the CVX toolbox within MATLAB to implement optimization.

http://cvxr.com/

Download, uncompress, set MATLAB to the cvx directory and do cvx_setup.

Take MATLAB back into your working directory.
Example of using CVX

```matlab
T = 150; N = 50;
R = randn(T, N);
rho = 0.02;
tau = 1;
mu = rand(N,1);

cvx_begin quiet
variable w(N)
    minimize( norm(rho*ones(T,1)-R*w) + tau*norm(w,1) )
subject to
    w'*ones(N,1) == 1;
    w'*mu == rho;
    w > 0;

cvx_end

figure(1), clf, bar(w); grid on
```

Note: Data random - probably won’t work all the time

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Portfolio weights: \( \mathbf{w} = [w_1 \ w_2 \ ... \ w_n]^T \)

Returns: \( \mathbf{a} \); \( E \ [\mathbf{a}] = \bar{\mathbf{a}} \); \( E \ [(\mathbf{a} - \bar{\mathbf{a}})(\mathbf{a} - \bar{\mathbf{a}})^T] = \Sigma \)
Portfolio weights: \( \mathbf{w} = [w_1 \ w_2 \ldots \ w_n]^T \)

Returns: \( \mathbf{a}; \quad E[\mathbf{a}] = \bar{\mathbf{a}}; \quad E[(\mathbf{a} - \bar{\mathbf{a}})(\mathbf{a} - \bar{\mathbf{a}})^T] = \Sigma \)

We consider an adjustment to the portfolio of value \( \mathbf{x} \)

New portfolio: \( \mathbf{w} + \mathbf{x} \); Wealth: \( \mathbf{a}^T (\mathbf{w} + \mathbf{x}) \)
• Portfolio weights: \( w = [w_1 \ w_2 \ldots w_n]^T \)

• Returns: \( a; \ E[a] = \bar{a}; \ E[(a - \bar{a})(a - \bar{a})^T] = \Sigma \)

• We consider an adjustment to the portfolio of value \( x \)

• New portfolio: \( w + x \); Wealth: \( a^T(w + x) \)

• Portfolio return and variance:

\[
E[W] = \bar{a}^T(w + x) \\
E[(W - E[W])^2] = (w + x)^T \Sigma (w + x)
\]
Portfolio weights: \( w = [w_1 \ w_2 \ \ldots \ w_n]^T \)

Returns: \( a; \ E [a] = \bar{a}; \ E [(a - \bar{a})(a - \bar{a})^T] = \Sigma \)

We consider an adjustment to the portfolio of value \( x \)

New portfolio: \( w + x \);  Wealth: \( a^T (w + x) \)

Portfolio return and variance:

\[
E [W] = \bar{a}^T (w + x)
\]
\[
E [(W - E [W])^2] = (w + x)^T \Sigma (w + x)
\]

Transaction cost: \( \phi (x) \)
Portfolio Optimization with Transaction Costs

- Portfolio weights: $\mathbf{w} = [w_1 \ w_2 \ldots \ w_n]^T$
- Returns: $\mathbf{a}; \ E[\mathbf{a}] = \bar{\mathbf{a}}; \ E[(\mathbf{a} - \bar{\mathbf{a}})(\mathbf{a} - \bar{\mathbf{a}})^T] = \Sigma$
- We consider an adjustment to the portfolio of value $\mathbf{x}$
- New portfolio: $\mathbf{w} + \mathbf{x}$; Wealth: $\mathbf{a}^T(\mathbf{w} + \mathbf{x})$
- Portfolio return and variance:
  
  $$E[\mathbf{W}] = \bar{\mathbf{a}}^T(\mathbf{w} + \mathbf{x})$$
  $$E[(\mathbf{W} - E[\mathbf{W}])^2] = (\mathbf{w} + \mathbf{x})^T \Sigma (\mathbf{w} + \mathbf{x})$$
  
- Transaction cost: $\phi(\mathbf{x})$
- Budget Constraint:
  
  $$1^T \mathbf{x} + \phi(\mathbf{x}) \leq 0$$
Possible Optimizations:
Possible Optimizations:

\[
\begin{align*}
\text{maximize} \quad & \mathbf{a}^T (\mathbf{w} + \mathbf{x}) \\
\text{subject to} \quad & \mathbf{1}^T \mathbf{x} + \phi(\mathbf{x}) \leq 0 \\
& \mathbf{w} + \mathbf{x} \in S
\end{align*}
\]

$S$ some feasible set (with other constraints)
Possible Optimizations:

\[
\begin{align*}
\text{maximize} \quad & \quad \bar{a}^T (w + x) \\
\text{subject to} \quad & \quad 1^T x + \phi(x) \leq 0 \\
& \quad w + x \in S
\end{align*}
\]

\(S\) some feasible set (with other constraints)

\[
\begin{align*}
\text{minimize} \quad & \quad \phi(x) \\
\text{subject to} \quad & \quad \bar{a}^T (w + x) \geq r_{\text{min}} \\
& \quad w + x \in S
\end{align*}
\]
Costs are separable (usual assumption):

\[ \phi(x) = \sum_{i=1}^{n} \phi_i(x_i) \]
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\[
\phi(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(x_i)
\]

\[
\phi_i(x_i) = \begin{cases} 
\alpha_i^+ x_i, & x_i \geq 0 \\
-\alpha_i^- x_i, & x_i \leq 0 
\end{cases}
\]

- \(\alpha_i^+\) and \(\alpha_i^-\) cost rates for buying and selling asset \(i\)
Lobo et al. (2007) (cont’d)
Modeling Transaction Costs

Costs are separable (usual assumption):

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\end{cases} \]

- \( \alpha_i^+ \) and \( \alpha_i^- \) cost rates for buying and selling asset \( i \)
- Convex cost function
- \( \phi_i = \alpha_i^+ x_i^+ + \alpha_i^- x_i^- \) with \( x_i^+ \geq 0 \) and \( x_i^- \geq 0 \)
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\( \alpha_i^+ \) and \( \alpha_i^- \) cost rates for buying and selling asset \( i \)
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\( \phi_i = \alpha_i^+ x_i^+ + \alpha_i^- x_i^- \) with \( x_i^+ \geq 0 \) and \( x_i^- \geq 0 \)

Fixed plus linear transaction costs:

\[ \phi_i(x_i) = \begin{cases} 
0, & x_i = 0 \\
\beta_i^+ + \alpha_i^+ x_i, & x_i \geq 0 \\
\beta_i^- - \alpha_i^- x_i, & x_i \leq 0 
\end{cases} \]
Lobo et al. (2007) (cont’d)
Modeling Transaction Costs

Costs are separable (usual assumption):

\[ \phi(x) = \sum_{i=1}^{n} \phi_i(x_i) \]

\[ \phi_i(x_i) = \begin{cases} 
\alpha_i^+ x_i, & x_i \geq 0 \\
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- \( \alpha_i^+ \) and \( \alpha_i^- \) cost rates for buying and selling asset \( i \)
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Fixed plus linear transaction costs:

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\end{cases} \]

- This is non-convex
Diversification Constraints

- Limit the amount of investment in any asset

\[ w_i + x_i \leq p_i, \quad i = 1, 2, ..., n \]
Diversification Constraints

- Limit the amount of investment in any asset
  \[ w_i + x_i \leq p_i, \quad i = 1, 2, \ldots, n \]

- Limit the fraction of total wealth held in each asset
  \[ w_i + x_i \leq 1^T (w + x) \]
Diversification Constraints

- Limit the amount of investment in any asset

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- Limit exposure in any small group of assets (say in a sector)

\[ \sum_{i=1}^{r} (w_i + x_i)[i] \leq 1^T (w + x) \]

(Tricky, but can show this is convex – see Eqn (11) in paper)
Constraints on short-selling
... on individual asset

\[ w_i + x_i \geq -s_i, \quad i = 1, \ldots, n \]
Constraints on short-selling

... on individual asset

\[ w_i + x_i \geq -s_i, \quad i = 1, \ldots, n \]

...or as bound on the total short position

\[ \sum_{i=1}^{n} (w_i + x_i) \leq S \]
Lobo et al. (2007) (cont’d)

- Constraints on short-selling
  ... on individual asset

\[ w_i + x_i \geq -s_i, \quad i = 1, \ldots, n \]

... or as bound on the total short position

\[ \sum_{i=1}^{n} (w_i + x_i)_- \leq S \]

- Collateralization:

\[ \sum_{i=1}^{n} (w_i + x_i)_- \leq \gamma \sum_{i=1}^{n} (w_i + x_i)_+ \]

*What I have borrowed to sell is smaller than a fraction of what I own*
Variance:

$$(w + x)^T \Sigma (w + x) \leq \sigma_{\text{max}}$$
Variance:

$$( w + x )^T \Sigma ( w + x ) \leq \sigma_{\text{max}}$$

Can also be written as:

$$\| \Sigma^{1/2} ( w + x ) \| \leq \sigma_{\text{max}}$$
Variance:

\[ (w + x)^T \Sigma (w + x) \leq \sigma_{\text{max}} \]

Can also be written as:

\[ \| \Sigma^{1/2} (w + x) \| \leq \sigma_{\text{max}} \]

- This is \textit{Second Order Cone} constraint.
Shortfall Risk Constraint:

\[ P \left( W \geq W^{\text{low}} \right) \geq \eta \]
Shortfall Risk Constraint:

$$P \left( W \geq W^{\text{low}} \right) \geq \eta$$

$$W = a^T (w + x) \sim \mathcal{N}(\mu, \sigma^2)$$
Shortfall Risk Constraint:

\[ P \left( W \geq W^{\text{low}} \right) \geq \eta \]

\[ W = a^T (w + x) \sim \mathcal{N}(\mu, \sigma^2) \]
Shortfall Risk Constraint:

\[ P \left( W \geq W^{\text{low}} \right) \geq \eta \]

\[ W = a^T (w + x) \sim \mathcal{N} (\mu, \sigma^2) \]
But $\frac{W - \mu}{\sigma} \sim \mathcal{N}(0, 1)$
But \((W - \mu)/\sigma \sim \mathcal{N}(0, 1)\)

Hence

\[
P \left( \frac{W - \mu}{\sigma} \leq \frac{W_{\text{low}} - \mu}{\sigma} \right) = \Phi \left( \frac{W_{\text{low}} - \mu}{\sigma} \right)
\]

\[
\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp \left\{ -\frac{t^2}{2} \right\} dt
\]

\[
\Phi^{-1}(1 - \eta) = \frac{W_{\text{low}} - \mu}{\sigma} = \Phi^{-1}(\eta)
\]

Using \(\mu = a^T (w + x)\) and \(\sigma^2 = (w + x)^T \Sigma (w + x)\)
But \((W - \mu)/\sigma \sim \mathcal{N}(0, 1)\)

Hence

\[
P\left(\frac{W - \mu}{\sigma} \leq \frac{W_{\text{low}} - \mu}{\sigma}\right) = \Phi\left(\frac{(W_{\text{low}} - \mu)}{\sigma}\right)
\]

\[
\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left\{\frac{-t^2}{2}\right\} dt
\]

\[
\frac{W_{\text{low}} - \mu}{\sigma} \leq \Phi^{-1}(1 - \eta)
\]
But \((W - \mu)/\sigma \sim \mathcal{N}(0, 1)\)

Hence

\[
P\left( \frac{W - \mu}{\sigma} \leq \frac{W^{\text{low}} - \mu}{\sigma} \right) = \Phi\left( \left( \frac{W^{\text{low}} - \mu}{\sigma} \right) \right)
\]

\[
\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp \left\{ -\frac{t^2}{2} \right\} \, dt
\]

\[
\frac{W^{\text{low}} - \mu}{\sigma} \leq \Phi^{-1}(1 - \eta)
\]

\[
\Phi^{-1}(1 - \eta) = -\Phi^{-1}(\eta)
\]

\[
\mu - W^{\text{low}} \geq \Phi^{-1}(\eta) \sigma
\]
But \((W - \mu)/\sigma \sim \mathcal{N}(0, 1)\)

Hence

\[
P\left(\frac{W - \mu}{\sigma} \leq \frac{W^{\text{low}} - \mu}{\sigma}\right) = \Phi\left(\frac{(W^{\text{low}} - \mu)}{\sigma}\right)
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\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left\{-\frac{t^2}{2}\right\} \, dt
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\frac{W^{\text{low}} - \mu}{\sigma} \leq \Phi^{-1}(1 - \eta)
\]

\[
\Phi^{-1}(1 - \eta) = -\Phi^{-1}(\eta)
\]

\[
\mu - W^{\text{low}} \geq \Phi^{-1}(\eta) \sigma
\]

Using \(\mu = \mathbf{a}^T(\mathbf{w} + \mathbf{x})\) and \(\sigma^2 = (\mathbf{w} + \mathbf{x})^T \Sigma (\mathbf{w} + \mathbf{x})\)

\[
\Phi^{-1}(\eta) \| \Sigma^{1/2}(\mathbf{w} + \mathbf{x}) \| \leq \mathbf{a}^T(\mathbf{w} + \mathbf{x}) - W^{\text{low}}
\]
\[ \Phi^{-1}(1-\eta) = -\Phi^{-1}(\eta) \]
Lobo et al. (2007) (cont’d)
Shortfall Risk on M-V Space
maximize $\bar{a}^T (w + x^+ - x^-)$

subject to $1^T (x^+ - x^-) + \sum_{i=1}^{n} \left( \alpha_i^+ x_i^+ + \alpha_i^- x_i^- \right) \leq 0$

$x_i^+ \geq 0, x_i^- \geq 0, i = 1, 2, \ldots, n$

$w_i + x_i^+ - x_i^- \geq s_i, i = 1, 2, \ldots, n$

$\Phi^{-1}(\eta_j) \| \Sigma^{1/2} (w + x^+ - x^-) \| \leq \bar{a}^T (w + x^+ - x^-) - W_j^{\text{low}}, j = 1, 2$
m1 = [0.15 0.2 0.08 0.1]’;
C1 = [ 0.2 0.05 -0.01 0.0
0.05 0.30 0.015 0.0
-0.01 0.015 0.10 0.0
0.0 0.0 0.0 0.0
];

m2 = [0.15 0.2 0.08]’;
C2 = [ 0.2 0.05 -0.01
0.05 0.30 0.015
-0.01 0.015 0.10
];

[V1, M1, PWts1] = NaiveMV(m1, C1, 25);
[V2, M2, PWts2] = NaiveMV(m2, C2, 25);

figure(2), clf,
plot(V1, M1, 'b', V2, M2, 'r', 'LineWidth', 3),
title('Mean Variance Portfolio', 'FontSize', 22)
xlabel('Portfolio Risk', 'FontSize',18)
ylabel('Portfolio Return', 'FontSize', 18);
Efficiency, no-arbitrage and fair price
Efficiency, no-arbitrage and fair price

Example:

- Price today $S(0)$
- $A$ and $B$ enter into a future contract to sell/buy at price $F$ at time $T$
Efficiency, no-arbitrage and fair price

Example:

- Price today $S(0)$
- $A$ and $B$ enter into a future contract to sell/buy at price $F$ at time $T$
- $A$ borrows $S(0)$ from the bank, buys the asset and waits till $T$
- At time $T$, $A$ owes the bank $S(0) \exp(rT)$ and has the asset to sell to $B$. 
Derivatives Pricing

- Efficiency, no-arbitrage and fair price

Example:

- Price today $S(0)$

- $A$ and $B$ enter into a future contract to sell/buy at price $F$ at time $T$

- $A$ borrows $S(0)$ from the bank, buys the asset and waits till $T$

- At time $T$, $A$ owes the bank $S(0) \exp(rT)$ and has the asset to sell to $B$.

- $F = S(0) \exp(rT)$, else arbitrage opportunity
Options

Call: right to buy at price $K$ at time $T$

Put: right to sell at price $K$ at time $T$

Exercise of contract
- European style: only at time $T$
- American style: any time in $0 \rightarrow T$
Options

- Call: right to buy at price $K$ at time $T$

Diagram:

\[ K \quad S(T) \]
Options

- **Call:** right to buy at price $K$ at time $T$

- **Put:** right to sell at price $K$ at time $T$
Options

- **Call**: right to buy at price $K$ at time $T$

- **Put**: right to sell at price $K$ at time $T$

**Exercise of contract**
- European style: only at time $T$
- American style: any time in $0 \rightarrow T$
Example: Put-Call Parity

- Portfolio $P_1$: European Call + cash $K \exp(-rT)$
Example: Put-Call Parity

- Portfolio $P_1$: European Call + cash $K \exp(-rT)$
- Portfolio $P_2$: European Put + one share of underlying stock
- Values at time $t = 0$
Example: Put-Call Parity

- Portfolio $P_1$: European Call + cash $K \exp(-rT)$
- Portfolio $P_2$: European Put + one share of underlying stock

Values at time $t = 0$

\[ P_1 \quad C + K \exp(-rT) \]
Example: Put-Call Parity

Portfolio $P_1$: European Call + cash $K \exp(-rT)$

Portfolio $P_2$: European Put + one share of underlying stock

Values at time $t = 0$

\[
\begin{align*}
P_1 & \quad C + K \exp(-rT) \\
P_2 & \quad P + S(0)
\end{align*}
\]
Example: Put-Call Parity

- Portfolio $P_1$: European Call + cash $K \exp(-rT)$
- Portfolio $P_2$: European Put + one share of underlying stock

Values at time $t = 0$

$$
P_1 = C + K \exp(-rT) \\
P_2 = P + S(0)
$$

Value of portfolios at time $t = T$

$$
S(T) > K \\
S(T) = K + P_1 - [S(T) - K] + K
$$

Both portfolios having the same value at time $t = T$ should also have the same value at $t = 0$.

$C + K \exp(-rT) = P + S(0)$
Example: Put-Call Parity

- Portfolio $P_1$: European Call + cash $K \exp(-rT)$
- Portfolio $P_2$: European Put + one share of underlying stock

Values at time $t = 0$

\[
\begin{align*}
& P_1 \quad C + K \exp(-rT) \\
& P_2 \quad P + S(0)
\end{align*}
\]

Value of portfolios at time $t = T$

\[
\begin{align*}
& S(T) > K \quad P_1 \quad [S(T) - K] + K = S(T) \\
& \quad P_2 \quad 0 + S(T) = S(T)
\end{align*}
\]
Example: Put-Call Parity

- **Portfolio** $P_1$: European Call + cash $K \exp(-rT)$
- **Portfolio** $P_2$: European Put + one share of underlying stock

Values at time $t = 0$

$$
\begin{align*}
P_1 & = C + K \exp(-rT) \\
P_2 & = P + S(0)
\end{align*}
$$

Value of portfolios at time $t = T$

$$
\begin{align*}
S(T) > K & \quad P_1 \quad [S(T) - K] + K = S(T) \\
S(T) < K & \quad P_1 \quad 0 + K = K \\
S(T) & \quad P_2 \quad 0 + S(T) = S(T)
\end{align*}
$$
Example: Put-Call Parity

- Portfolio $P_1$: European Call + cash $K \exp(-rT)$
- Portfolio $P_2$: European Put + one share of underlying stock

Values at time $t = 0$

$$
\begin{align*}
P_1 & \quad C + K \exp(-rT) \\
P_2 & \quad P + S(0)
\end{align*}
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Value of portfolios at time $t = T$

$$
\begin{align*}
S(T) > K & \quad P_1 \quad [S(T) - K] + K = S(T) \\
P_2 & \quad 0 + S(T) = S(T) \\
S(T) < K & \quad P_1 \quad 0 + K = K \\
P_2 & \quad [K - S(T)] + S(T) = K
\end{align*}
$$
Example: Put-Call Parity

- Portfolio $P_1$: European Call + cash $K \exp(-rT)$
- Portfolio $P_2$: European Put + one share of underlying stock

Values at time $t = 0$

\[
P_1 = C + K \exp(-rT) \\
P_2 = P + S(0)
\]

Value of portfolios at time $t = T$

- $S(T) > K$
  \[
P_1 = [S(T) - K] + K = S(T) \\
P_2 = 0 + S(T) = S(T)
\]
- $S(T) < K$
  \[
P_1 = 0 + K = K \\
P_2 = [K - S(T)] + S(T) = K
\]

Both portfolios having the same value at time $t = T$ should also have the same value at $t = 0$.

\[
C + K \exp(-rT) = P + S(0)
\]
Geometric Brownian motion for stock price

\[
\begin{align*}
    dS(t) &= \mu S(t) dt + \sigma S(t) dW(t) \\
    \frac{dS(t)}{S(t)} &= \mu dt + \sigma dW(t)
\end{align*}
\]
- Geometric Brownian motion for stock price

\[
dS(t) = \mu S(t)dt + \sigma S(t)dW(t)
\]

\[
\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)
\]

- Stochastic differential equation for the log of the process

\[
F(S, t) = \log S(t)
\]
- Geometric Brownian motion for stock price

\[ dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t) \]

\[ \frac{dS(t)}{S(t)} = \mu \,dt + \sigma \,dW(t) \]

- Stochastic differential equation for the log of the process

\[ F(S, t) = \log S(t) \]

- Ito’s lemma tells us about increments \( dF \)

- Terms needed to apply Ito’s lemma

\[
\begin{align*}
\frac{\partial F}{\partial t} &= 0 \\
\frac{\partial F}{\partial S} &= 1 \\
\frac{\partial^2 F}{\partial S^2} &= -\frac{1}{S^2}
\end{align*}
\]
\[ dF = \left( \frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW \]
\[ dF = \left( \frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW \]

\[ = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW \]

\[ \log S(t) = \log S(0) + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma dW(t) \]
\[ dF = \left( \frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW \]

\[ = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW \]

\[
\log S(t) = \log S(0) + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma dW(t)
\]

- \( dW(t) = \epsilon \sqrt{t} \) where \( \epsilon \sim \mathcal{N}(0, 1) \)

\[
\log S(t) \sim \mathcal{N} \left[ \log S(0) + \left( \mu - \frac{\sigma^2}{2} \right) t, \ \sigma^2 t \right]
\]

- Log of asset price has a normal distribution
\[ dF = \left( \frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW \]

\[ = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW \]

\[
\log S(t) = \log S(0) + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma dW(t)
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- \(dW(t) = \epsilon \sqrt{t}\) where \(\epsilon \sim \mathcal{N}(0, 1)\)

\[
\log S(t) \sim \mathcal{N} \left[ \log S(0) + \left( \mu - \frac{\sigma^2}{2} \right) t, \ \sigma^2 t \right]
\]

- Log of asset price has a normal distribution
- Also

\[
S(t) = S(0) \exp \left( (\mu - \sigma^2/2)t + \sigma \sqrt{t} \epsilon \right)
\]
Black-Scholes Model

- Model

\[ dS(t) = \mu S(t) dt + \sigma S(t) dW(t) \]
Black-Scholes Model

- Model

\[ dS(t) = \mu S(t) dt + \sigma S(t) dW(t) \]

- Change in option price

\[ df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} dt \]
Black-Scholes Model

- Model
  \[ dS(t) = \mu S(t) dt + \sigma S(t) dW(t) \]

- Change in option price
  \[ df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} dt \]

- At maturity
  \[ f(S(T), T) = \max\{S(T) - K, 0\} \]
Black-Scholes Model

- **Model**

\[ dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \]

- **Change in option price**

\[ df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} dt \]

- **At maturity**

\[ f(S(T), T) = \max\{S(T) - K, 0\} \]

- **Consider a portfolio**
  - Own \( \Delta \) stocks (long)
  - One call option sold

\[ \Pi = \Delta S - f(S, t) \]
\[ d\Pi = \Delta dS - df \]
\[ = \left( \Delta - \frac{\partial f}{\partial S} \right) dS - \left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt \]
\[ d\Pi = \Delta dS - df \]
\[ = \left( \Delta - \frac{\partial f}{\partial S} \right) dS - \left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt \]

- Term in \( dS \) (stochastic) can be eliminated by choosing \( \Delta \)

\[ \Delta = \frac{\partial f}{\partial S} \]
\[ d\Pi = \Delta dS - df \]
\[ = \left( \Delta - \frac{\partial f}{\partial S} \right) dS - \left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt \]

- Term in \( dS \) (stochastic) can be eliminated by choosing \( \Delta \)
  \[ \Delta = \frac{\partial f}{\partial S} \]

- With this choice of \( \Delta \) (balance between short and long), the portfolio is riskless.
- \( d\Pi = r\Pi dt \)
\[ d\Pi = \Delta dS - df \]
\[ = \left( \Delta - \frac{\partial f}{\partial S} \right) dS - \left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt \]

- Term in \( dS \) (stochastic) can be eliminated by choosing \( \Delta \)

\[ \Delta = \frac{\partial f}{\partial S} \]

- With this choice of \( \Delta \) (balance between short and long), the portfolio is riskless.
- \( d\Pi = r \Pi dt \)
- Eliminating \( d\Pi \)

\[ \left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt = r \left( f - S \frac{\partial f}{\partial S} \right) dt \]
\[ \frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0 \]
Partial differential equation
\[ \frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0 \]

Boundary condition
- European Call: \( f(S, T) = \max\{S - K, 0\} \)
- European Put: \( f(S, T) = \max\{K - S, 0\} \)
Partial differential equation

\[ \frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0 \]

Boundary condition

- European Call: \( f(S, T) = \max\{S - K, 0\} \)
- European Put: \( f(S, T) = \max\{K - S, 0\} \)

Black-Scholes

\[ C = S_0 \mathcal{N}(d_1) - K \exp(-rT)\mathcal{N}(d_2) \]

\[ d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}} \]

\[ d_2 = d_1 - \sigma \sqrt{T} \]

\[ \mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-y^2/2)dy \]
Partial differential equation

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0$$

Boundary condition

- European Call: \( f(S, T) = \max\{S - K, 0\} \)
- European Put: \( f(S, T) = \max\{K - S, 0\} \)

Black-Scholes

\[
C = S_0 N(d_1) - K \exp(-rT)N(d_2)
\]

\[
d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}
\]

\[
d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}
\]

\[
d_2 = d_1 - \sigma\sqrt{T}
\]

\[
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-y^2/2)dy
\]

Put-Call parity

\[
P = K \exp(-rT)N(-d_2) - S_0N(-d_1)
\]
Binomial Lattice

$S_0, f_0, uS_0, f_u, dS_0, f_d$
Construct a portfolio:
- A riskless bond, initial price $B_0 = 1$ and future value $B_1 = \exp(r\delta t)$
- Underlying asset, initial value $S_0$
- Number of stocks $\Delta$, number of bonds $\Psi$

Initial value of this portfolio $\Pi_0 = \Delta S_0 + \Psi$

Future value depends on price movement up or down:
- $\Pi_u = \Delta S_0 u + \Psi \exp(r\delta t)$
- $\Pi_d = \Delta S_0 d + \Psi \exp(r\delta t)$

We can solve for a portfolio that will replicate option payoff
$\Delta S_0 u + \Psi \exp(r\delta t) = f_u$
$\Delta S_0 d + \Psi \exp(r\delta t) = f_d$

... and solve for $\Delta$ and $\Psi$
Options Pricing on a Binomial Model

- Construct a portfolio:
  - A riskless bond, initial price $B_0 = 1$ and future value $B_1 = \exp(r\delta t)$
  - Underlying asset, initial value $S_0$
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\[ \Pi_0 = \Delta S_0 + \Psi \]
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  \[
  \begin{cases}
  \Pi_u = \Delta S_0 u + \Psi \exp(r\delta t) \\
  \Pi_d = \Delta S_0 d + \Psi \exp(r\delta t)
  \end{cases}
  \]
Options Pricing on a Binomial Model

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  \Pi_u = \Delta S_0 u + \Psi \exp(r\delta t) \\
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  \end{cases} \]

- We can solve for a portfolio that will replicate option payoff
  \[ \Delta S_0 u + \Psi \exp(r\delta t) = f_u \]
  \[ \Delta S_0 d + \Psi \exp(r\delta t) = f_d \]

and solve for $\Delta$ and $\Psi$.
... solving

\[ \Delta = \frac{f_u - f_d}{S_0(u - d)} \]

\[ \Psi = \exp(-r\delta t) \frac{uf_d - df_u}{u - d} \]
... solving

\[ \Delta = \frac{f_u - f_d}{S_0(u - d)} \]

\[ \psi = \exp(-r\delta t) \frac{uf_d - df_u}{u - d} \]

No arbitrage \( \implies \) initial value of this portfolio should be \( f_0 \)

\[ f_0 = \Delta S_0 + \psi \]
\[ = \frac{f_u - f_d}{(u - d)} + \exp(-r\delta t) \frac{uf_d - df_u}{u - d} \]
\[ = \exp(-r\delta t) \left\{ \frac{f_u}{u - d} + \frac{f_d}{u - d} \right\} \]
... solving

\[ \Delta = \frac{f_u - f_d}{S_0(u - d)} \]
\[ \psi = \exp(-r\delta t) \frac{uf_d - df_u}{u - d} \]

No arbitrage \( \implies \) initial value of this portfolio should be \( f_0 \)

\[ f_0 = \Delta S_0 + \psi \]
\[ = \frac{f_u - f_d}{(u - d)} + \exp(-r\delta t) \frac{uf_d - df_u}{u - d} \]
\[ = \exp(-r\delta t) \left\{ \frac{--}{u - d} f_u + \frac{--}{u - d} f_d / \right\} \]

Defining probabilities

\[ \pi_u = \frac{\exp(r\delta t) - d}{u - d} \text{ and } \pi_d = \frac{u - \exp(r\delta t)}{u - d} \]

option price interpreted as discounted expected value

\[ f_0 = \exp(-r\delta t) (\pi_u f_u + \pi_d f_d) \]
Binomial Lattice

\[
\begin{align*}
S & \quad Su \quad Su^2 d \\
S d & \quad Su d^2 \\
S d^3 & \quad Su^3 \\
S & \quad S \\
S d & \quad Su d^2
\end{align*}
\]
Calibrating a Binomial Lattice

- When are these equivalent?

\[ dS = rS \, dt + \sigma S \, dW \]
Calibrating a Binomial Lattice

- When are these equivalent?

\[ dS = rS dt + \sigma S dW \]

- Log normal distribution

\[ \log (S_{t+\delta t}) \sim \mathcal{N} ((r - \sigma^2/2), \sigma^2 \delta t) \]
Calibrating a Binomial Lattice

- When are these equivalent?

\[ dS = r \, S \, dt + \sigma \, S \, dW \]

- Log normal distribution

\[ \log (S_{t+\delta t}) \sim \mathcal{N} \left( (r - \sigma^2/2), \sigma^2 \delta t \right) \]

- Mean and variance of log normal distribution

(log of the variable is normal, what is mean and variance of the variable?)

\[
E[S_{t+\delta t}] = \exp(r \, \delta t) \\
\text{Var}[S_{t+\delta t}] = \exp(2r\delta t) \left( \exp(\sigma^2\delta t) - 1 \right)
\]
Mean for the lattice

\[ E[S_t + \delta t] = p u S_t + (1 - p) d S_t \]

Equating the means...

\[ p u S_t + (1 - p) d S_t = \exp(r \delta t) S_t \]

Variance on the lattice

\[ \text{Var}[S_t + \delta t] = E[S_t^2 + \delta t] - E^2[S_t + \delta t] \]

\[ = S_t^2 (p u^2 + (1 - p) d^2) - S_t^2 \exp(2r \delta t) \]

... which from the dynamical model is...

\[ \text{Var}[S_t + \delta t] = S_t^2 \exp(2r \delta t) \left( \exp(\sigma^2 \delta t) - 1 \right) \]
Mean for the lattice

\[ E [S_{t+\delta}] = p u S_t + (1 - p) d S_t \]
Mean for the lattice

\[ E [S_{t+\delta t}] = p u S_t + (1 - p) d S_t \]

Equating the means...

\[ p u S_t + (1 - p) d S_t = \exp(r \delta t) S_t \]

\[ p = \frac{\exp(r \delta t) - d}{u - d} \]
Calibrating binomial lattice (cont’d)

- Mean for the lattice
  \[ E[S_{t+\delta t}] = puS_t + (1-p)dS_t \]

- Equating the means...
  \[ puS_t + (1-p)dS_t = \exp(r\delta t)S_t \]
  \[ p = \frac{\exp(r\delta t) - d}{u - d} \]

- Variance on the lattice
  \[ \text{Var}[S_{t+\delta t}] = E[S_{t+\delta t}^2] - E^2[S_{t+\delta t}] \]
  \[ = S_t^2 (pu^2 + (1-p)d^2) - S_t^2 \exp(2r\delta t) \]
Calibrating binomial lattice (cont’d)

- Mean for the lattice

\[ E[S_{t+\delta t}] = puS_t + (1 - p)dS_t \]

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\[ puS_t + (1 - p)dS_t = \exp(r \delta t) S_t \]

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\[ \text{Var}[S_{t+\delta t}] = E[S_{t+\delta t}^2] - E^2[S_{t+\delta t}] \]
\[ = S_t^2 (pu^2 + (1 - p)d^2) - S_t^2 \exp(2r\delta t) \]

... which from the dynamical model is...

\[ \text{Var}[S_{t+\delta t}] = S_t^2 \exp(2r\delta t) (\exp(\sigma^2 \delta t) - 1) \]
Equating the two variances

\[ S_t^2 \exp(2r\delta t) (\exp(\sigma^2 \delta t) - 1) = S_t^2 \left( pu^2 + (1 - p)d^2 \right) - S_t^2 \exp(2r\delta t) \]
Equating the two variances

\[ S_t^2 \exp(2r\delta t) (\exp(\sigma^2\delta t) - 1) = S_t^2 (pu^2 + (1 - p)d^2) - S_t^2 \exp(2r\delta t) \]

Which reduces to

\[ \exp(2r\delta t + \sigma^2\delta t) = pu^2 + (1 - p)d^2 \]
Equating the two variances

\[ S_t^2 \exp(2r\delta t) (\exp(\sigma^2\delta t) - 1) = S_t^2 \left( pu^2 + (1 - p)d^2 \right) - S_t^2 \exp(2r\delta t) \]

Which reduces to

\[ \exp(2r\delta t + \sigma^2\delta t) = pu^2 + (1 - p)d^2 \]

Substitute for \( p \) and simplify

\[ \exp(2r\delta t + \sigma^2\delta t) = (u + d) \exp(r\delta t) - 1 \]

... and because \( u = 1/d \),

\[ u^2 \exp(r\delta t) - u \left( 1 + \exp(2r\delta t + \sigma^2\delta t) \right) + \exp(r\delta t) = 0 \]

... a quadratic equation in \( u \).
\[ u = \frac{(1 + \exp(2r\delta t + \sigma^2\delta t)) + \sqrt{(1 + \exp(2r\delta t + \sigma^2\delta t)^2 - 4\exp(2r\delta t)}}{2\exp(r\delta t)} \]

Taylor series expansion of \( \exp(x) \)

\[ (1 + \exp(2r\delta t + \sigma^2\delta t))^2 - 4\exp(2r\delta t) \approx (2 + (2r + \sigma^2)\delta t)^2 - 4(1 + 2r\delta t) \approx 4\sigma^2\delta t \]

\[ u \approx \frac{2 + (2r + \sigma^2)\delta t + 2\sigma\sqrt{\delta t}}{2\exp(r\delta t)} \]

\[ \approx \left(1 + r\delta t + \frac{\sigma^2}{2}\delta t + \sigma\sqrt{\delta t}\right)(1 - r\delta t) \]

\[ \approx 1 + r\delta t + \frac{\sigma^2}{2}\delta t + \sigma\sqrt{\delta t} - r\delta t \]

\[ = 1 + \sigma\sqrt{\delta t} + \frac{\sigma^2}{2}\delta t \]
Calibrating the Binomial Lattice (cont’d)

\[ u = \exp(\sigma \sqrt{\delta t}) \]

\[ d = \exp(-\sigma \sqrt{\delta t}) \]

\[ p = \frac{\exp(r\delta t) - d}{u - d} \]

\[ dS = rS \, dt + \sigma S \, dW \]
European call option; $S_0 = K = 50$; $r = 0.1$; $\sigma = 0.4$; maturity in five months.

```plaintext
>> call = blsprice(50, 50, 0.1, 5/12, 0.4)
call =
   6.1165
```

We can now build the lattice

| $\delta t$ | $1/12$ | $0.0833$ |
| $u$        | $\exp(\sigma \sqrt{t})$ | $1.1224$ |
| $d$        | $1/u$ | $0.8909$ |
| $p$        | $(\exp(r\delta t) - d) / (u - d)$ | $0.5073$ |
function [price, lattice] = LatticeEurCall(S0,K,r,T,sigma,N)

deltaT = T/N;
u=exp(sigma * sqrt(deltaT));
d=1/u;
p=(exp(r*deltaT) - d)/(u-d);
lattice = zeros(N+1,N+1);

for i=0:N
    lattice(i+1,N+1)=max(0 , S0*(u^i)*(d^(N-i)) - K);
end

for j=N-1:-1:0
    for i=0:j
        lattice(i+1,j+1) = exp(-r*deltaT) * ...
            (p * lattice(i+2,j+2) + (1-p) * lattice(i+1,j+2));
    end
end

price = lattice(1,1);
function price = AmPutLattice(S0,K,r,T,sigma,N)
deltaT = T/N;
u=exp(sigma * sqrt(deltaT));
d=1/u;
p=(exp(r*deltaT) - d)/(u-d);
discount = exp(-r*deltaT);
p_u = discount*p;
p_d = discount*(1-p);
SVals = zeros(2*N+1,1);
SVals(N+1) = S0;

[...]
function price = AmPutLattice(S0,K,r,T,sigma,N)

[...] 

for i=1:N 
    SVals(N+1+i) = u*SVals(N+i);
    SVals(N+1-i) = d*SVals(N+2-i);
end
PVals = zeros(2*N+1,1);
for i=1:2:2*N+1
    PVals(i) = max(K-SVals(i),0);
end

[...]
function price = AmPutLattice(S0,K,r,T,sigma,N)

[...]
for tau=1:N
    for i= (tau+1):2:(2*N+1-tau)
        hold = p_u*PVals(i+1) + p_d*PVals(i-1);
        PVals(i) = max(hold, K-SVals(i));
    end
end
price = PVals(N+1);

- Decisions at every point during backtracking

\[ f_{i,j} = \max \{ K - S_{i,j}, \exp(-r\delta t)(p f_{i+1,j+1} + (1 - p) f_{i,j+1}) \} \]
We will look at inference as expectations...

\[ E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) \, dx \]
We will look at inference as expectations...

\[ E [g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) \, dx \]

Consider the integral

\[ I = \int_{0}^{1} g(x) \, dx \]
We will look at inference as expectations...

\[ E [g(X)] = \int_{-\infty}^{+\infty} g(x)f_X(x) \, dx \]

Consider the integral

\[ I = \int_{0}^{1} g(x) \, dx \]

Think of this as computing the expected value (of a function of a uniform random variable):

\[ E [g(U)] , \text{ where } U \sim (0, 1) \]
We will look at inference as expectations...

\[
E [g(X)] = \int_{-\infty}^{+\infty} g(x) f_x(x) \, dx
\]

Consider the integral

\[
I = \int_{0}^{1} g(x) \, dx
\]

Think of this as computing the expected value (of a function of a uniform random variable):

\[
E [g(U)], \quad \text{where } U \sim (0, 1)
\]

We approximate the integral by

\[
\hat{I}_m = \frac{1}{m} \sum_{i=1}^{m} g(U_i)
\]
Where will we use this?
- Where will we use this?
- European call option

\[ f = \exp(-rT) E[f_T] \]
Where will we use this?

- European call option

\[ f = \exp(-rT) E[f_T] \]

- \( f_T \) is payoff at maturity \( T \); fair price is discounted expected payoff
Where will we use this?

European call option

\[ f = \exp(-rT) E[f_T] \]

\( f_T \) is payoff at maturity \( T \); fair price is discounted expected payoff

\[ f_T = \max\left\{ 0, \ S(0) \exp((r - \sigma^2/2) T + \sigma \sqrt{T} \epsilon) - K \right\} \]
- European call option

\[ f = \exp(-rT) \mathbb{E}[f_T] \]

- \( f_T \) is payoff at maturity \( T \); fair price is discounted expected payoff

\[ f_T = \max\left\{ 0, \ S(0) \exp((r - \sigma^2/2)T + \sigma\sqrt{T}\epsilon) - K \right\} \]

---

% BlsMC1.m

```matlab
function Price = BlsMC1(S0,K,r,T,sigma,NRepl)
    nuT = (r - 0.5*sigma^2)*T;
    siT = sigma * sqrt(T);
    DiscPayoff = exp(-r*T)*max(0, S0*exp(nuT+siT*randn(NRepl,1))-K);
    Price = mean(DiscPayoff);
```

```matlab
> S0=50; K=60; r=0.05; T=1; sigma=0.2;
> randn('state', 0);
> BlsMC1(S0, K, r, T, sigma, 1000)
```

ans =

\[ 1.2562 \]
Where will we use this?

European call option

\[ f = \exp(-rT) E[f_T] \]

- \( f_T \) is payoff at maturity \( T \); fair price is discounted expected payoff
- \( f_T = \max\{0, S(0) \exp((r - \sigma^2/2)T + \sigma\sqrt{T}\epsilon) - K\} \)

---

```matlab
function Price = BlsMC1(S0,K,r,T,sigma,NRepl)

nuT = (r - 0.5*sigma^2)*T;
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DiscPayoff = exp(-r*T)*max(0, S0*exp(nuT+siT*randn(NRepl,1))-K);
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ans = 1.2562
```
Is this a good approach?

Different answers on different runs

S0=50; K=60; r=0.05; T=1; sigma=0.2;
randn('state', 0);
BlsMC1(S0, K, r, T, sigma, 1000)
ans = 1.2562
BlsMC1(S0, K, r, T, sigma, 1000)
ans = 1.8783
BlsMC1(S0, K, r, T, sigma, 1000)
ans = 1.7864

What if we had large number of samples?
BlsMC1(S0, K, r, T, sigma, 1000000)
ans = 1.6295
BlsMC1(S0, K, r, T, sigma, 1000000)
ans = 1.6164
BlsMC1(S0, K, r, T, sigma, 1000000)
ans = 1.6141
Is this a good approach?

Different answers on different runs
> S0=50; K=60; r=0.05; T=1; sigma=0.2;
> randn('state', 0);
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
    1.2562
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
    1.8783
> BlsMC1(S0, K, r, T, sigma, 1000)
ans =
    1.7864
Is this a good approach?

- Different answers on different runs
  ```matlab
  S0=50; K=60; r=0.05; T=1; sigma=0.2;
  randn('state', 0);
  BlsMC1(S0, K, r, T, sigma, 1000)
  ans =
      1.2562
  BlsMC1(S0, K, r, T, sigma, 1000)
  ans =
      1.8783
  BlsMC1(S0, K, r, T, sigma, 1000)
  ans =
      1.7864
  ```

- What if we had large number of samples?
  ```matlab
  BlsMC1(S0, K, r, T, sigma, 1000000)
  ans =
      1.6295
  BlsMC1(S0, K, r, T, sigma, 1000000)
  ans =
      1.6164
  BlsMC1(S0, K, r, T, sigma, 1000000)
  ans =
      1.6141
  ```
Sampling: Inverse Transform

Sample $X$ from $f(x)$; Cumulative distribution $F(x)$

- Draw $U \sim U(0,1)$
- Return $X = F^{-1}(U)$

Example: Exponential distribution

$X \sim \text{exp}(\mu)$

Cumulative $F(x) = 1 - \exp(-\mu x)$

Inverse $x = -\frac{1}{\mu} \log(1 - U)$

Distributions of $U$ and $(1-U)$ are the same

Hence return: $-\log(U)/\mu$
Sampling: Inverse Transform

Sample $X$ from $f(x)$; Cumulative distribution $F(x)$

- Draw $U \sim U(0, 1)$
- Return $X = F^{-1}(U)$

\[
P\{X \leq x\} = P\{F^{-1}(U) \leq x\} = P\{U \leq F(x)\} = F(x)
\]
Sampling: Inverse Transform

- Sample \( X \) from \( f(x) \); Cumulative distribution \( F(x) \)

  - Draw \( U \sim U(0,1) \)
  - Return \( X = F^{-1}(U) \)

\[
P\{X \leq x\} = P\{F^{-1}(U) \leq x\} = P\{U \leq F(x)\} = F(x)
\]

- Example: Exponential distribution \( X \sim \exp(\mu) \)
- Cumulative
  \[
  F(x) = 1 - \exp(-\mu x)
  \]
- Inverse
  \[
  x = -\frac{1}{\mu} \log(1 - U)
  \]
- Distributions of \( U \) and \((1 - U)\) are the same
  Hence return: \(- \log(U)/\mu\)
Sampling: Acceptance-Rejection Method

- Probability density function: \( f(x) \)

Consider a known function \( t(x) \), such that \( t(x) \geq f(x) \), \( \forall x \in I \)

\( I \) is the support for \( f \) (region in which it is defined)

\( t(x) \) is a probability density of normalized \( r(x) = t(x) / c \)

\[ c = \int_I t(x) \, dx \]

Generate \( Y \sim r \)

Generate \( U \sim U(0, 1) \)

If \( U \leq f(Y) / t(Y) \) return \( X = Y \)

Else go to 1

On average \( c \) trials to accept a sample.
Probability density function: $f(x)$

Consider a known function $t(x)$, such that

$$t(x) \geq f(x), \ \forall x \in I$$
Sampling: Acceptance-Rejection Method

- Probability density function: $f(x)$
- Consider a known function $t(x)$, such that
  \[ t(x) \geq f(x), \quad \forall x \in \mathcal{I} \]

- $\mathcal{I}$ is the support for $f$ (region in which it is defined)
Probability density function: $f(x)$

Consider a known function $t(x)$, such that

$$t(x) \geq f(x), \quad \forall x \in \mathcal{I}$$

$I$ is the support for $f$ (region in which it is defined)

$t(x)$ is a probability density of normalized

$$r(x) = \frac{t(x)}{c} \quad c = \int_{\mathcal{I}} t(x) \, dx$$
Sampling: Acceptance-Rejection Method

- Probability density function: $f(x)$
- Consider a known function $t(x)$, such that
  $$t(x) \geq f(x), \ \forall x \in \mathcal{I}$$

- $\mathcal{I}$ is the support for $f$ (region in which it is defined)
- $t(x)$ is a probability density of normalized

  $$r(x) = \frac{t(x)}{c} \quad c = \int_{\mathcal{I}} t(x) \, dx$$

1. Generate $Y \sim r$
2. Generate $U \sim U(0, 1)$
3. If $U \leq \frac{f(Y)}{t(Y)}$ return
   $X = Y$
   Else go to 1
Sampling: Acceptance-Rejection Method

- Probability density function: \( f(x) \)
- Consider a known function \( t(x) \), such that \( t(x) \geq f(x) \), \( \forall x \in \mathcal{I} \)

\( \mathcal{I} \) is the support for \( f \) (region in which it is defined)
\( t(x) \) is a probability density of normalized

\[
r(x) = \frac{t(x)}{c} \quad c = \int_{\mathcal{I}} t(x) \, dx
\]

1. Generate \( Y \sim r \)
2. Generate \( U \sim U(0,1) \)
3. If \( U \leq f(Y)/t(Y) \) return \( X = Y \)
   Else go to 1
\[ f(x) = 30(x^2 - 2x^3 + x^4), \quad x \in [0, 1] \]
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Algorithm

1. Draw \( U_1 \) and \( U_2 \)
2. If \( U_2 \leq 16(U_1^2 - 2U_1^3 + U_1^4) \) accept \( X = U_1 \)
   Else
   go to 1


- $f(x) = 30(x^2 - 2x^3 + x^4)$, $x \in [0, 1]$
- Algorithm
  1. Draw $U_1$ and $U_2$
  2. If $U_2 \leq 16(U_1^2 - 2U_1^3 + U_1^4)$
     accept $X = U_1$
     Else
     go to 1
- Exercise:
  - Draw the graph of $f(x)$
  - Simulate 1000 samples using above algorithm
  - Draw a histogram to the same scale as $f(x)$ – do they match? Is it better with 100000 samples?
  - On average, how many trials were needed through the accept-reject loop for each sample?
Variance Reduction

- Independent samples $X_i$
- Sample mean (estimates mean $\mu = E[X_i]$ from $n$ samples)

$$\bar{X}(n) = \frac{1}{n} \sum_{i=1}^{n} X_i$$
Variance Reduction

- Independent samples $X_i$
- Sample mean (estimates mean $\mu = E [X_i]$ from $n$ samples)

$$\bar{X}(n) = \frac{1}{n} \sum_{i=1}^{n} X_i$$

- Sample variance

$$S^2(n) = \frac{1}{(n-1)} \sum_{i=1}^{n} \left[ X_i - \bar{X}(n) \right]^2$$

Two points:
- More samples $n$ reduces the variance in estimation
- Variance reduction schemes can control $\sigma^2$
Variance Reduction

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  \]

- Error of the estimator
  \[
  E \left[ (\overline{X}(n) - \mu)^2 \right] = \text{Var} \left[ \overline{X}(n) \right]
  \]
  \[
  = \text{Var} \left[ \frac{1}{n} \sum_{i=1}^{n} X_i \right]
  \]
  \[
  = \frac{1}{n^2} \times n \times \sigma^2 = \frac{\sigma^2}{n}
  \]
Variance Reduction

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- Error of the estimator

$$E[(\bar{X}(n) - \mu)^2] = \text{Var}[\bar{X}(n)]$$

$$= \text{Var}\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n^2} \times n \times \sigma^2 = \frac{\sigma^2}{n}$$

- Two points:
  - More samples $n$ reduces the variance in estimation
  - Variance reduction schemes can control $\sigma^2$
Variance reduction: Antithetic Sampling

- Pair of sequences

\[
\begin{pmatrix}
X_1^{(1)} & X_1^{(2)} & \ldots & X_1^n \\
X_2^{(1)} & X_2^{(2)} & \ldots & X_2^n
\end{pmatrix}
\]

- Columns (horizontally) are independent
- \(X_1^{(i)}\) and \(X_2^{(i)}\) are dependent.
Variance reduction: Antithetic Sampling

- Pair of sequences

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Variance reduction: Antithetic Sampling

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- Columns (horizontally) are independent
- \(X_1^{(i)}\) and \(X_2^{(i)}\) are dependent.
- Sample is a function of each pair: \(X^{(i)} = \left( X_1^{(i)} + X_2^{(i)} \right) / 2\)
- Variance
  \[
  \text{Var} \left[ \bar{X}(n) \right] = \frac{1}{n} \text{Var} \left[ X^{(i)} \right]
  = \frac{1}{4n} \left\{ \text{Var}(X_1^{(i)}) + \text{Var}(X_2^{(i)}) + 2 \text{Cov}(X_1^{(i)}, X_2^{(i)}) \right\}
  = \frac{1}{2n} \text{Var}(X) (1 + \rho)
  \]

- Uniform random number \(\{U_k\}\) and \(\{1 - U_k\}\) as sequences.
function [Price, CI] = BlsMC2(S0,K,r,T,sigma,NRepl)
  nuT = (r - 0.5*sigma^2)*T;
  siT = sigma * sqrt(T);
  DiscPayoff = exp(-r*T)*max(0, S0*exp(nuT+siT*randn(NRepl,1))-K);
  [Price, VarPrice, CI] = normfit(DiscPayoff);

function [Price, CI] = BlsMCAV(S0,K,r,T,sigma,NPairs)
  nuT = (r - 0.5*sigma^2)*T;
  siT = sigma * sqrt(T);
  Veps = randn(NPairs,1);
  Payoff1 = max( 0 , S0*exp(nuT+siT*Veps) - K);
  Payoff2 = max( 0 , S0*exp(nuT+siT*(-Veps)) - K);
  DiscPayoff = exp(-r*T) * 0.5 * (Payoff1+Payoff2);
  [Price, VarPrice, CI] = normfit(DiscPayoff);
function [Price, CI] = BlsMC2(S0,K,r,T,sigma,NRepl)
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  [Price, VarPrice, CI] = normfit(DiscPayoff);
Homework
Test the two functions: BlsMC and BlsMCAV

(Brandimarte, p248)

```matlab
> randn('state', 0)
> [Price, CI] = BlsMC2(50,50,0.05,1,0.4,200000)
Price=
    9.0843
CI =
    9.0154
    9.1532

pause
> (CI(2)-CI(1))/Price
ans =
    0.0152

pause
> randn('state', 0)
> [Price, CI] = BlsMCAV(50,50,0.05,1,0.4,200000)
Price=
    9.0553
CI =
    8.9987
    9.1118

pause
> (CI(2)-CI(1))/Price
ans =
    0.0125
```
We have seen three tools for pricing options
- Closed form Black-Scholes
- Binomial lattice
- Monte Carlo

\[ V_t = \sum_{j=1}^{J} \lambda_j \phi_j(x) + w^T x + w_0 \]
Approximating option prices with a neural network

- We have seen three tools for pricing options
  - Closed form Black-Scholes
  - Binomial lattice
  - Monte Carlo

- How well can the relationship between asset price and option price be approximated?

Approximating option prices with a neural network

- We have seen three tools for pricing options
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- How well can the relationship between asset price and option price be approximated?


\[
x = [S/X \ (T - t)]^T
\]

\[
c = \sum_{j=1}^{J} \lambda_j \phi_j(x) + w^T x + w_0
\]
Figure 4. Simulated call option prices normalized by strike price and plotted versus
\[
C/X = -0.06 \sqrt{\left[ \frac{S/X - 1.35}{T - t - 0.45} \right] \cdot \left[ \begin{array}{cc} 59.79 & -0.03 \\ -0.03 & 10.24 \end{array} \right] \left[ \frac{S/X - 1.35}{T - t - 0.45} \right] + 2.55} \\
- 0.03 \sqrt{\left[ \frac{S/X - 1.18}{T - t - 0.24} \right] \cdot \left[ \begin{array}{cc} 59.79 & -0.03 \\ -0.03 & 10.24 \end{array} \right] \left[ \frac{S/X - 1.18}{T - t - 0.24} \right] + 1.97} \\
+ 0.03 \sqrt{\left[ \frac{S/X - 0.98}{T - t + 0.20} \right] \cdot \left[ \begin{array}{cc} 59.79 & -0.03 \\ -0.03 & 10.24 \end{array} \right] \left[ \frac{S/X - 0.98}{T - t + 0.20} \right] + 0.00} \\
+ 0.10 \sqrt{\left[ \frac{S/X - 1.05}{T - t + 0.10} \right] \cdot \left[ \begin{array}{cc} 59.79 & -0.03 \\ -0.03 & 10.24 \end{array} \right] \left[ \frac{S/X - 1.05}{T - t + 0.10} \right] + 1.62} \\
+ 0.14 S/X - 0.24 (T - t) - 0.01. 
\] (9)