Learning Activities and Assignment

ELEC2210: Applied Electromagnetics
ELEC2211: Electromechanical Energy Conversion
ELEC2219: Electromagnetism for EEE

Pt 2 Electrical Eng, Mechatronic Eng, EEE
Structure

- Lectures
  - 10 weeks (~30h joint between ELEC2210, ELEC2211 and ELEC2219)
  - 2 weeks separately
- Tutorials / Examples
  - Each week 1h
- Assessments
  - 50% Exam (2h) + 35% Coursework + 15% Laboratories
- Coursework
  - TAS+FD (analytical + COMSOL) 10%
  - Finite Elements (using Magnet) 25%
- Laboratories
  - 3 experiments
Review of electromagnetic fields

ELEC2210: Applied Electromagnetics
ELEC2211: Electromechanical Energy Conversion
ELEC2219: Electromagnetism for EEE

Pt 2 Electrical Eng, Mechatronic Eng, EEE
Revision topics

- Historical overview
- Basic properties of electric and magnetic forces
- Dimensions, units, notations
- Effects of materials
- Statics
Examples of EM Applications

- Astronomy
- The Very Large Array of Radio Telescopes
- Global Positioning System (GPS)
- Motor
- Plasma propulsion
- Radar
- Optical fiber
- Telecommunication
- Electromagnetic sensors
- Cell phone
- Microwave ablation for liver cancer treatment
### Table 1-1: Fundamental SI units.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Unit</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>meter</td>
<td>m</td>
</tr>
<tr>
<td>Mass</td>
<td>kilogram</td>
<td>kg</td>
</tr>
<tr>
<td>Time</td>
<td>second</td>
<td>s</td>
</tr>
<tr>
<td>Electric Current</td>
<td>ampere</td>
<td>A</td>
</tr>
<tr>
<td>Temperature</td>
<td>kelvin</td>
<td>K</td>
</tr>
<tr>
<td>Amount of substance</td>
<td>mole</td>
<td>mol</td>
</tr>
</tbody>
</table>

### Table 1-2: Multiple and submultiple prefixes.

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Symbol</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>exa</td>
<td>E</td>
<td>10^{18}</td>
</tr>
<tr>
<td>peta</td>
<td>P</td>
<td>10^{15}</td>
</tr>
<tr>
<td>tera</td>
<td>T</td>
<td>10^{12}</td>
</tr>
<tr>
<td>giga</td>
<td>G</td>
<td>10^{9}</td>
</tr>
<tr>
<td>mega</td>
<td>M</td>
<td>10^{6}</td>
</tr>
<tr>
<td>kilo</td>
<td>k</td>
<td>10^{3}</td>
</tr>
<tr>
<td>milli</td>
<td>m</td>
<td>10^{-3}</td>
</tr>
<tr>
<td>micro</td>
<td>μ</td>
<td>10^{-6}</td>
</tr>
<tr>
<td>nano</td>
<td>n</td>
<td>10^{-9}</td>
</tr>
<tr>
<td>pico</td>
<td>p</td>
<td>10^{-12}</td>
</tr>
<tr>
<td>femto</td>
<td>f</td>
<td>10^{-15}</td>
</tr>
<tr>
<td>atto</td>
<td>a</td>
<td>10^{-18}</td>
</tr>
</tbody>
</table>
Electromagnetics in the Classical Era

1827 Georg Simon Ohm (German) formulates Ohm's law relating electric potential to current and resistance.

1827 Joseph Henry (American) introduces the concept of inductance, and builds one of the earliest electric motors. He also assisted Samuel Morse in the development of the telegraph.

1831 Michael Faraday (English) discovers that a changing magnetic flux can induce an electromotive force.

1835 Carl Friedrich Gauss (German) formulates Gauss's law relating the electric flux flowing through an enclosed surface to the enclosed electric charge.

1873 James Clerk Maxwell (Scottish) publishes his Treatise on Electricity and Magnetism in which he unites the discoveries of Coulomb, Oersted, Ampère, Faraday, and others into four elegantly constructed mathematical equations, now known as Maxwell's Equations.

1887 Heinrich Hertz (German) builds a system that can generate electromagnetic waves (at radio frequencies) and detect them.

1888 Nikola Tesla (Croatian-American) invents the ac (alternating current) electric motor.

1895 Wilhelm Röntgen (German) discovers X-rays. One of his first X-ray images was of the bones in his wife's hands. [1901 Nobel prize in physics.]

1897 Joseph John Thomson (English) discovers the electron and measures its charge-to-mass ratio. [1906 Nobel prize in physics.]

1905 Albert Einstein (German-American) explains the photoelectric effect discovered earlier by Hertz in 1887. [1921 Nobel prize in physics.]
Telecommunications

1923
Vladimir Zworykin (Russian-American) invents television. In 1926, John Baird (Scottish) transmits TV images over telephone wires from London to Glasgow. Regular TV broadcasting began in Germany (1935), England (1936), and the United States (1939).

1926
Transatlantic telephone service between London and New York.

1932
First microwave telephone link, installed (by Marconi) between Vatican City and the Pope’s summer residence.

1933
Edwin Armstrong (American) invents frequency modulation (FM) for radio transmission.

1935
Robert Watson-Watt (Scottish) invents radar.

1938
H. A. Reeves (American) invents pulse code modulation (PCM).

1947
William Shockley, Walter Brattain, and John Bardeen (all Americans) invent the junction transistor at Bell Labs. [1956 Nobel prize in physics.]

1958
Jack Kilby (American) builds first integrated circuit (IC) on germanium and, independently, Robert Noyce (American) builds first IC on silicon.

1960
Echo, the first passive communication satellite is launched, and successfully reflects radio signals back to Earth. In 1963, the first communication satellite is placed in geosynchronous orbit.

1969
ARPANET is established by the U.S. Department of Defense, to evolve later into the Internet.

1979
Japan builds the first cellular telephone network:
- 1990 electronic beepers become common.
- 1995 cell phones become widely available.
- 2002 cell phone supports video and Internet.

1984
Worldwide Internet becomes operational.

1988
First transatlantic optical fiber cable between the U.S. and Europe.

1997
Mars Pathfinder sends images to Earth.

1955
Pager is introduced as a radio communication product in hospitals and factories.

1955
Narinder Kapany (Indian-American) demonstrates the optical fiber as a low-loss, light-transmission medium.

2004
Wireless communication supported by many airports, university campuses, and other facilities.

2012
Smartphones worldwide exceed 1 billion.
Fundamental Forces of Nature

- The **nuclear force**, which is the strongest of the four, but its range is limited to **subatomic scales**, such as nuclei.

- The **electromagnetic force**, which exists between all charged particles. It is the dominant force in **microscopic** systems, such as atoms and molecules, and its strength is on the order of $10^{-2}$ that of the nuclear force.

- The **weak-interaction force**, whose strength is only $10^{-14}$ that of the nuclear force. Its primary role is in interactions involving certain radioactive elementary particles.

- The **gravitational force** is the weakest of all four forces, having a strength on the order of $10^{-41}$ that of the nuclear force. However, it often is the dominant force in **macroscopic** systems, such as the solar system.
Charge: Electrical property of particles

Units: coulomb

One coulomb: amount of charge accumulated in one second by a current of one ampere.

1 coulomb represents the charge on $\sim 6.241 \times 10^{18}$ electrons

The coulomb is named for a French physicist, Charles-Augustin de Coulomb (1736-1806), who was the first to measure accurately the forces exerted between electric charges.

Charge of an electron

$$e = 1.602 \times 10^{-19} \text{ C}$$

Charge conservation

Cannot create or destroy charge, only transfer
Coulomb’s experiments demonstrated that:

(1) two like charges repel one another, whereas two charges of opposite polarity attract,

(2) the force acts along the line joining the charges, and

(3) its strength is proportional to the product of the magnitudes of the two charges and inversely proportional to the square of the distance between them.

\[
F_{e_{21}} = \hat{R}_{12} \frac{q_1 q_2}{4\pi \varepsilon_0 R_{12}^2} \quad \text{(N)} \quad \text{(in free space)},
\]

Force exerted on charge 2 by charge 1
If any point charge \( q' \) is present in an electric field \( \mathbf{E} \) (due to other charges), the point charge will experience a force acting on it equal to \( \mathbf{F}_c = q' \mathbf{E} \).

\[
\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi \varepsilon_0 R^2} \quad \text{(V/m)} \quad \text{(in free space)},
\]

Permittivity of free space

Electric field lines

\[ \hat{\mathbf{R}} \]
Electric Field Inside Dielectric Medium

Polarization of atoms changes electric field

New field can be accounted for by changing the permittivity

\[ E = \hat{R} \frac{q}{4\pi \varepsilon R^2} \quad \text{(V/m)} \]

Permittivity of the material

Another quantity used in EM is the electric flux density \( \mathbf{D} \):

\[ \mathbf{D} = \varepsilon \mathbf{E} \quad \text{(C/m}^2\text{)} \]
Magnetic Field

Electric charges can be isolated, but magnetic poles always exist in pairs.

Magnetic field induced by a current in a long wire

\[ \mathbf{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r} \]  (T)

\[ \mu_0 = 4\pi \times 10^{-7} \text{ henry per meter (H/m)} \]

Magnetic permeability of free space

Electric and magnetic fields are connected through the speed of light:

\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ (m/s)} \]
Static vs. Dynamic

Static conditions:
charges are stationary or moving, but if moving, they do so at a constant velocity

Table 1-3: The three branches of electromagnetics.

<table>
<thead>
<tr>
<th>Branch</th>
<th>Condition</th>
<th>Field Quantities (Units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrostatics</td>
<td>Stationary charges $(\partial q/\partial t = 0)$</td>
<td>Electric field intensity $E$ (V/m)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Electric flux density $D$ (C/m$^2$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$D = \varepsilon E$</td>
</tr>
<tr>
<td>Magnetostatics</td>
<td>Steady currents $(\partial I/\partial t = 0)$</td>
<td>Magnetic flux density $B$ (T)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Magnetic field intensity $H$ (A/m)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$B = \mu H$</td>
</tr>
<tr>
<td>Dynamics</td>
<td>Time-varying currents $(\partial I/\partial t \neq 0)$</td>
<td>$E$, $D$, $B$, and $H$</td>
</tr>
<tr>
<td>(Time-varying fields)</td>
<td></td>
<td>$(E, D)$ coupled to $(B, H)$</td>
</tr>
</tbody>
</table>

Under static conditions, electric and magnetic fields are independent, but under dynamic conditions, they become coupled.
Material Properties

Table 1-4: Constitutive parameters of materials.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
<th>Free-space Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical permittivity $\varepsilon$</td>
<td>F/m</td>
<td>$\varepsilon_0 = 8.854 \times 10^{-12}$ (F/m)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\simeq \frac{1}{36\pi} \times 10^{-9}$ (F/m)</td>
</tr>
<tr>
<td>Magnetic permeability $\mu$</td>
<td>H/m</td>
<td>$\mu_0 = 4\pi \times 10^{-7}$ (H/m)</td>
</tr>
<tr>
<td>Conductivity $\sigma$</td>
<td>S/m</td>
<td>0</td>
</tr>
</tbody>
</table>
Electrostatics and Steady Currents

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Pt 2 Electrical Eng, Mechatronic Eng, EEE
Electrostatics and Steady Currents

- Electric fields
  - Potential
  - Etc.

- Some Maths
  - Vector Analysis

- Steady Currents
  - Concept of flux and Charge conservation
  - Analytical methods
  - Approximate methods of analysis

- Electrostatics
  - Differential form of Gauss’s law
  - Capacitors
Electric Scalar Potential

The term “voltage” is short for “voltage potential” and synonymous with electric potential.

Minimum force needed to move charge against E field:

\[ F_{\text{ext}} = -F_c = -qE. \]  

The work done, or energy expended, in moving any object a vector differential distance \( dl \) while exerting a force \( F_{\text{ext}} \) is

\[ dW = F_{\text{ext}} \cdot dl = -qE \cdot dl \]  

Work, or energy, is measured in joules (J). If the charge is moved a distance \( dy \) along \( \hat{y} \), then

\[ dW = -q(-\hat{y}E) \cdot \hat{y} dy = qE dy. \]

The differential electric potential energy \( dW \) per unit charge is called the differential electric potential (or differential voltage) \( dV \). That is,

\[ dV = \frac{dW}{q} = -E \cdot dl \]  

Compare with gravity!
Electric Scalar Potential

Figure 4-12: In electrostatics, the potential difference between $P_2$ and $P_1$ is the same irrespective of the path used for calculating the line integral of the electric field between them.

\[
\int_{P_1}^{P_2} dV = - \int_{P_1}^{P_2} E \cdot dl,
\]

\[
V_{21} = V_2 - V_1 = - \int_{P_1}^{P_2} E \cdot dl, \tag{4.39}
\]

\[
\int_C E \cdot dl = 0 \quad \text{(Electrostatics).} \tag{4.40}
\]

Compare with gravity!

A vector field whose line integral along any closed path is zero is called a conservative or an irrotational field. Hence, the electrostatic field $E$ is conservative.
Electrostatics and Steady Currents

- Electric fields
- Some Maths
  - Vector Analysis
- Steady Currents
- Electrostatics
Vector Analysis

- Vectors
- Scalar Product
- Gradient
Differential length vector

\[ \mathbf{d}l = \hat{x} \, dl_x + \hat{y} \, dl_y + \hat{z} \, dl_z = \hat{x} \, dx + \hat{y} \, dy + \hat{z} \, dz, \quad (3.34) \]

where \( dl_x = dx \) is a differential length along \( \hat{x} \), and similar interpretations apply to \( dl_y = dy \) and \( dl_z = dz \).

Differential area vectors

\[ ds_x = \hat{x} \, dl_y \, dl_z = \hat{x} \, dy \, dz \quad (y-z \text{ plane}), \quad (3.35a) \]

with the subscript on \( ds \) denoting its direction. Similarly,

\[ ds_y = \hat{y} \, dx \, dz \quad (x-z \text{ plane}), \quad (3.35b) \]
\[ ds_z = \hat{z} \, dx \, dy \quad (x-y \text{ plane}). \quad (3.35c) \]

A differential volume equals the product of all three differential lengths:

\[ dV = dx \, dy \, dz. \quad (3.36) \]
Laws of Vector Algebra

\[ \mathbf{A} = \hat{\mathbf{a}} |\mathbf{A}| = \hat{\mathbf{a}} \mathbf{A} \]

\[ \mathbf{A} = \hat{x} A_x + \hat{y} A_y + \hat{z} A_z \]

\[ A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \]

\[ \hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{x} A_x + \hat{y} A_y + \hat{z} A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \]
Equality of Two Vectors

\[ \mathbf{A} = \mathbf{a} \mathbf{A} = \mathbf{\hat{x}} A_x + \mathbf{\hat{y}} A_y + \mathbf{\hat{z}} A_z, \quad (3.6a) \]
\[ \mathbf{B} = \mathbf{b} \mathbf{B} = \mathbf{\hat{x}} B_x + \mathbf{\hat{y}} B_y + \mathbf{\hat{z}} B_z, \quad (3.6b) \]

then \( \mathbf{A} = \mathbf{B} \) if and only if \( A = B \) and \( \mathbf{a} = \mathbf{b} \), which requires that \( A_x = B_x \), \( A_y = B_y \), and \( A_z = B_z \).

Equality of two vectors does not necessarily imply that they are identical; in Cartesian coordinates, two displaced parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if they lie on top of one another.

![Diagram](image)

**Figure 3-3:** Vector addition by (a) the parallelogram rule and (b) the head-to-tail rule.
Vector Multiplication: Scalar Product or "Dot Product"

\[ \mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB} \]

\[ A = |A| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \]

\[ \theta_{AB} = \cos^{-1} \left[ \frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt{\mathbf{A} \cdot \mathbf{A}} \sqrt{\mathbf{B} \cdot \mathbf{B}}} \right] \]

\[ \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1, \]
\[ \hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0. \]

**Figure 3-5:** The angle \( \theta_{AB} \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \), measured from \( \mathbf{A} \) to \( \mathbf{B} \) between vector tails. The dot product is positive if \( 0 \leq \theta_{AB} < 90^\circ \), as in (a), and it is negative if \( 90^\circ < \theta_{AB} \leq 180^\circ \), as in (b).

\[ \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \text{(commutative property)} \]
\[ \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad \text{(distributive property)} \]

Hence:

\[ \mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \]
From differential calculus, the temperature difference between points $P_1$ and $P_2$, $dT = T_2 - T_1$, is

$$dT = \frac{\partial T}{\partial x} \, dx + \frac{\partial T}{\partial y} \, dy + \frac{\partial T}{\partial z} \, dz. \quad (3.70)$$

Because $dx = \hat{x} \cdot dl$, $dy = \hat{y} \cdot dl$, and $dz = \hat{z} \cdot dl$, Eq. (3.70) can be rewritten as

$$dT = \hat{x} \frac{\partial T}{\partial x} \cdot dl + \hat{y} \frac{\partial T}{\partial y} \cdot dl + \hat{z} \frac{\partial T}{\partial z} \cdot dl$$

$$= \left[ \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right] \cdot dl. \quad (3.71)$$

$$\nabla T = \text{grad } T = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z}. \quad (3.72)$$

Equation (3.71) can then be expressed as

$$dT = \nabla T \cdot dl. \quad (3.73)$$

The symbol $\nabla$ is called the \textit{del} or \textit{gradient operator} and is

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \quad \text{(Cartesian)}. \quad (3.74)$$

**Figure 3-19:** Differential distance vector $dl$ between points $P_1$ defined as $P_2 = (x + dx, y + dy, z + dz)$ and $P_2 = (x, y, z)$. 

---

**Note:** This image contains a diagram illustrating a gradient vector field. The gradient vector at point $P_1$ is shown in the direction of the greatest rate of increase of the scalar field $T$. The vector is directed along the line of steepest ascent, pointing from $P_1$ to $P_2$. The gradient vector is perpendicular to the contour lines of the field, which represent constant values of $T$. The magnitude of the gradient vector indicates the steepness of the field. This diagram helps visualize how the gradient operator is used to calculate the rate of change of a scalar field in three dimensions.
With $dl = \hat{a}_l dl$, where $\hat{a}_l$ is the unit vector of $dl$, the **directional derivative** of $T$ along $\hat{a}_l$ is

$$\frac{dT}{dl} = \nabla T \cdot \hat{a}_l. \quad (3.75)$$

We can find the difference $(T_2 - T_1)$, where $T_1 = T(x_1, y_1, z_1)$ and $T_2 = T(x_2, y_2, z_2)$ are the values of $T$ at points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ not necessarily infinitesimally close to one another, by integrating both sides of Eq. (3.73). Thus,

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot dl. \quad (3.76)$$
Gradient (cont.)

Example 3-9: Directional Derivative

Find the directional derivative of \( T = x^2 + y^2 z \) along direction \( \hat{x}2 + \hat{y}3 - \hat{z}2 \) and evaluate it at \((1, -1, 2)\).

Solution: First, we find the gradient of \( T \):

\[
\nabla T = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) (x^2 + y^2 z) = \hat{x}2x + \hat{y}2yz + \hat{z}y^2.
\]

We denote \( \mathbf{l} \) as the given direction,

\[
\mathbf{l} = \hat{x}2 + \hat{y}3 - \hat{z}2.
\]

Its unit vector is

\[
\hat{a}_l = \frac{1}{|\mathbf{l}|} = \frac{\hat{x}2 + \hat{y}3 - \hat{z}2}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{\hat{x}2 + \hat{y}3 - \hat{z}2}{\sqrt{17}}.
\]

Application of Eq. (3.75) gives

\[
\frac{dT}{dl} = \nabla T \cdot \hat{a}_l = (\hat{x}2x + \hat{y}2yz + \hat{z}y^2) \cdot \left( \frac{\hat{x}2 + \hat{y}3 - \hat{z}2}{\sqrt{17}} \right)
\]

\[
= \frac{4x + 6yz - 2y^2}{\sqrt{17}}.
\]

At \((1, -1, 2)\),

\[
\frac{dT}{dl} \bigg|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = \frac{-10}{\sqrt{17}}.
\]
Relating $\mathbf{E}$ to $V$

\[ dV = -\mathbf{E} \cdot d\mathbf{l}. \quad (4.49) \]

For a scalar function $V$, Eq. (3.73) gives

\[ dV = \nabla V \cdot d\mathbf{l}, \quad (4.50) \]

where $\nabla V$ is the gradient of $V$. Comparison of Eq. (4.49) with Eq. (4.50) leads to

\[ \mathbf{E} = -\nabla V. \quad (4.51) \]

This differential relationship between $V$ and $\mathbf{E}$ allows us to determine $\mathbf{E}$ for any charge distribution by first calculating $V$ and then taking the negative gradient of $V$ to find $\mathbf{E}$. 
Microscopic quantities

- Let’s look inside materials!

- Elementary charges
  - Stationary
  - Motion
Charge Distributions

Volume charge density:

\[ \rho_v = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV} \quad \text{(C/m}^3\text{)} \]

Total Charge in a Volume

\[ Q = \int_V \rho_v \, dV \quad \text{(C)} \]

Surface and Line Charge Densities

\[ \rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad \text{(C/m}^2\text{)} \]

\[ \rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad \text{(C/m)} \]
Current Density

The amount of charge that crosses the tube’s cross-sectional surface $\Delta s'$ in time $\Delta t$ is therefore

$$\Delta q' = \rho_v \Delta V = \rho_v \Delta l \Delta s' = \rho_v u \Delta s' \Delta t. \quad (4.8)$$

For a surface with any orientation:

$$\Delta q = \rho_v u \cdot \Delta s \Delta t, \quad (4.9)$$

where $\Delta s = \hat{n} \Delta s$ and the corresponding total current flowing in the tube is

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v u \cdot \Delta s = J \cdot \Delta s, \quad (4.10)$$

where

$$J = \rho_v u \quad (\text{A/m}^2) \quad (4.11)$$

$J$ is called the current density.

Figure 4-2: Charges with velocity $u$ moving through a cross section $\Delta s'$ in (a) and $\Delta s$ in (b).

When a current is due to the actual movement of electrically charged matter, it is called a convection current, and $J$ is called a convection current density.
Conduction Current

The conductivity of a material is a measure of how easily electrons can travel through the material under the influence of an externally applied electric field.

Conduction current density:

\[ J = \sigma E \quad \text{(A/m}^2\text{)} \quad \text{(Ohm's law),} \]

A perfect dielectric is a material with \( \sigma = 0 \). In contrast, a perfect conductor is a material with \( \sigma = \infty \). Some materials, called superconductors, exhibit such a behavior.

Table 4-1: Conductivity of some common materials at 20°C.

<table>
<thead>
<tr>
<th>Material</th>
<th>Conductivity, ( \sigma ) (S/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conductors</strong></td>
<td></td>
</tr>
<tr>
<td>Silver</td>
<td>( 6.2 \times 10^7 )</td>
</tr>
<tr>
<td>Copper</td>
<td>( 5.8 \times 10^7 )</td>
</tr>
<tr>
<td>Gold</td>
<td>( 4.1 \times 10^7 )</td>
</tr>
<tr>
<td>Aluminum</td>
<td>( 3.5 \times 10^7 )</td>
</tr>
<tr>
<td>Iron</td>
<td>( 10^7 )</td>
</tr>
<tr>
<td>Mercury</td>
<td>( 10^6 )</td>
</tr>
<tr>
<td>Carbon</td>
<td>( 3 \times 10^4 )</td>
</tr>
<tr>
<td><strong>Semiconductors</strong></td>
<td></td>
</tr>
<tr>
<td>Pure germanium</td>
<td>2.2</td>
</tr>
<tr>
<td>Pure silicon</td>
<td>( 4.4 \times 10^{-4} )</td>
</tr>
<tr>
<td><strong>Insulators</strong></td>
<td></td>
</tr>
<tr>
<td>Glass</td>
<td>( 10^{-12} )</td>
</tr>
<tr>
<td>Paraffin</td>
<td>( 10^{-15} )</td>
</tr>
<tr>
<td>Mica</td>
<td>( 10^{-15} )</td>
</tr>
<tr>
<td>Fused quartz</td>
<td>( 10^{-17} )</td>
</tr>
</tbody>
</table>

Note how wide the range is, over 24 orders of magnitude.
Electrostatics and Steady Currents

- Electric fields
- Some Maths
- Steady Currents
- Electrostatics
Nomenclature - Steady Currents

- **E** – electric field, **J** – current density,
- **V** – voltage, **I** – current, **R** - resistance
- **ρ** - electric resistivity, **σ** - electric conductivity
- **v** – volume, **s** or **S** – area, **l** - length
Resistivity

For any conductor:

$$R = \frac{V}{I} = \frac{-\int_{l} E \cdot dl}{\int_{s} J \cdot ds} = \frac{-\int_{l} E \cdot dl}{\int_{s} \sigma E \cdot ds}.$$  

$$\Omega = \frac{l}{\sigma A}.$$ 

Using Eq. (4.63), the current flowing through the cross section $A$ at $x_2$ is

$$I = \int_{A} J \cdot ds = \int_{A} \sigma E \cdot ds = \sigma A E \quad (A). \quad (4.69)$$

From $R = V/I$, the ratio of Eq. (4.68) to Eq. (4.69) gives

$$R = \frac{l}{\sigma A}. \quad (4.70)$$

Longitudinal Resistor

$$V = V_1 - V_2 = -\int_{x_2}^{x_1} E \cdot dl$$

$$= -\int_{x_2}^{x_1} \hat{x} E_x \cdot \hat{x} dl = E_x l \quad (V). \quad (4.68)$$
Conductivity

\[ \sigma = -\rho_{ve}\mu_e + \rho_{vh}\mu_h \]
\[ = (N_e\mu_e + N_h\mu_h)e \quad \text{(S/m)} \quad \text{(semiconductor)}, \]
\[ (4.67a) \]

and its unit is siemens per meter (S/m). For a good conductor, \( N_h\mu_h \ll N_e\mu_e \), and Eq. (4.67a) reduces to

\[ \sigma = -\rho_{ve}\mu_e = N_e\mu_e e \quad \text{(S/m)} \quad \text{(conductor).} \]
\[ (4.67b) \]

In view of Eq. (4.66), in a perfect dielectric with \( \sigma = 0 \), \( \mathbf{J} = 0 \) regardless of \( \mathbf{E} \), and in a perfect conductor with \( \sigma = \infty \), \( \mathbf{E} = \mathbf{J}/\sigma = 0 \) regardless of \( \mathbf{J} \).

That is,

- Perfect dielectric: \( \mathbf{J} = 0 \),
- Perfect conductor: \( \mathbf{E} = 0 \).

\[ \rho_{ve} = \text{volume charge density of electrons} \]
\[ \rho_{he} = \text{volume charge density of holes} \]
\[ \mu_e = \text{electron mobility} \]
\[ \mu_h = \text{hole mobility} \]
\[ N_e = \text{number of electrons per unit volume} \]
\[ N_h = \text{number of holes per unit volume} \]
Example 4-8: Conduction Current in a Copper Wire

A 2-mm-diameter copper wire with conductivity of $5.8 \times 10^7$ S/m and electron mobility of 0.0032 (m²/V·s) is subjected to an electric field of 20 (mV/m). Find (a) the volume charge density of the free electrons, (b) the current density, (c) the current flowing in the wire, (d) the electron drift velocity, and (e) the volume density of the free electrons.

Solution:

(a)

$$\rho_{ve} = -\frac{\sigma}{\mu_e} = -\frac{5.8 \times 10^7}{0.0032} = -1.81 \times 10^{10} \text{ (C/m}^3).$$

(b)

$$J = \sigma E = 5.8 \times 10^7 \times 20 \times 10^{-3} = 1.16 \times 10^6 \text{ (A/m}^2).$$

(c)

$$I = JA$$

$$= J \left(\frac{\pi d^2}{4}\right) = 1.16 \times 10^6 \left(\frac{\pi \times 4 \times 10^{-6}}{4}\right) = 3.64 \text{ A.}$$

(d)

$$u_e = -\mu_e E = -0.0032 \times 20 \times 10^{-3} = -6.4 \times 10^{-5} \text{ m/s.}$$

The minus sign indicates that $u_e$ is in the opposite direction of $E$.

(e)

$$N_e = -\frac{\rho_{ve}}{e} = \frac{1.81 \times 10^{10}}{1.6 \times 10^{-19}} = 1.13 \times 10^{29} \text{ electrons/m}^3.$$
More complex shapes?

- Conductor - Composite block
  - Blue – Cu, Grey - graphite
- Connected to a battery
  - Terminals on the left and right edges
- Can we calculate the resistance?
- Also can we found the trajectories of electrons (current flow lines)?
Flow of steady current

\[ V = RI \quad \text{Ohm's law} \]

For a simply shaped resistor:

\[ R = \frac{l}{\sigma S} \]

\[ V = \frac{l}{\sigma S} I \]

\[ \sigma \frac{V}{l} = \frac{I}{S} \]

\[ \sigma E = J \]
A unit block to use?

Tubes and Slices

\[ R = \frac{l}{S} \rho \]
Concept of tubes and slices

- **tubes**
  - (thin insulating sheets)

- **slices**
  - (thin ‘superconducting’ sheets)
Calculations of Resistance

**m conductors in parallel:**
\[
\frac{1}{R} = \sum \frac{1}{r_i}
\]

\[
r_i = \frac{l}{\sigma \frac{S}{m}} = \frac{m l}{\sigma S}
\]

\[
\frac{1}{R} = \sum_{i=1}^{m} \frac{\sigma S}{m l} = \frac{\sigma S}{m l} \sum_{i=1}^{m} 1 = \frac{\sigma S}{l} \frac{m}{m}
\]

\[
\therefore \quad R = \frac{l}{\sigma S}
\]

**n conductors in series:**
\[
R = \sum r_z
\]

\[
r_z = \frac{l}{\sigma S}
\]

\[
R = \sum_{i=1}^{n} \frac{n}{\sigma S} = \frac{l}{\sigma S} \frac{n}{n}
\]

\[
\therefore \quad R = \frac{l}{\sigma S}
\]
Irregular Shape
Example

Calculations

$R^+ = \rho \frac{2a}{a} = 2\rho \quad \text{per unit depth}$

$R^- = \rho \frac{a}{a} + \rho \frac{a}{2a} = 1.5\rho \quad \text{per unit depth}$
Averaging

Example

\[ R^+ = \rho \frac{2a}{a} = 2 \rho \quad \text{per unit depth (tubes)} \]

\[ R^- = \rho \frac{a}{a} + \rho \frac{a}{2a} = 1.5 \rho \quad \text{per unit depth (slices)} \]

\[ R_{\text{ave}} = \frac{R^+ + R^-}{2} = \frac{2 + 1.5}{2} \rho = 1.75 \rho \quad \text{per unit depth} \]

Answer: \( R = 1.75(\pm0.25)\rho \quad \text{per unit depth (guaranteed !)} \)

Accurate result (from a finite element program): \( R = 1.73 \rho \quad \text{per unit depth} \)
Lower and Upper Band

- If tubes and slices are in correct position
  - there is negligible effect as they are not disturbing the field

- If they are not in correct position:
  - tubes → increase resistance giving $R^+$ upper bound
  - slices → decrease resistance giving $R^-$ lower bound

- Only in undisturbed field $R^- = R^+$

- otherwise $R^- < R < R^+$

Why? Think about minimum power dissipation in natural systems!
Joule’s Law

Consider: \( \mathbf{E} \cdot \mathbf{J} \) unit of W/m\(^3\) → power per unit volume, defined by \( \delta \mathbf{l} \cdot \delta \mathbf{S} \)

\[ \delta P = \mathbf{E} \cdot \mathbf{J} = \sigma E^2 = \frac{1}{\sigma} J^2 \]

Total power:

\[ P = \iiint \sigma E^2 \, dv = \frac{V^2}{R} \quad \text{or:} \quad P = \iiint \frac{1}{\sigma} J^2 \, dv = I^2 R \]

The power dissipated in a volume containing electric field \( \mathbf{E} \) and current density \( \mathbf{J} \) is:

\[ P = \int \int \int \mathbf{E} \cdot \mathbf{J} \, dV \quad \text{(W)} \quad \text{(Joule’s law)} \]

For a resistor, Joule’s law reduces to:

\[ P = I^2 R \quad \text{(W)} \]
Lower and Upper Band

- Power loss in tubes & in slices always greater than the minimum!

- $R^+ > R$

- $R^- < R$

**Tubes:**

\[ \text{Power} = I^2 R^+ \quad \text{where} \quad R^+ = \frac{1}{\sum_{i=1}^{t} \sum_{j=1}^{r} r_{ij}} \]

**Slices:**

\[ \text{Power} = \frac{V^2}{R^-} \quad \text{where} \quad R^- = \sum_{i=1}^{t} \sum_{j=1}^{r} \frac{1}{r_{ij}} \]

\[ r = \frac{l}{S \rho} \]

\(S\)
Example

Tubes and Slices

\[ R_{ave} = \frac{R^+ + R^-}{2} \]
Simple calculations

Tubes:
\[ R^+ = \frac{10}{3} = 3.33 \]

Slices:
\[ R^- = \frac{6}{5} + \frac{1}{3} + \frac{3}{5} = 2.13 \]

\[ R_{\text{average}} = 2.73 \times \rho \quad (\text{per unit depth}) \]
Complex Shapes

What is the real shape of tubes and slices?
Current Funneling

Surface: Electric potential (V)  Streamline: Current density
Replace insulator with another conductor

Surface: Electric potential (V)  Streamline: Current density

How can we find current density and potential distribution?
Flow of Steady Current

**Current tubes**: since current is not a vector it is convenient to use the current density as a vector in the direction of the normal to the cross-section:

\[ \delta I = \mathbf{J} \cdot \hat{n} \, dS = \mathbf{J} \cdot \delta \hat{S} \]

Ohm's Law \( \sigma \mathbf{E} = \mathbf{J} \) links the voltage gradient to the current density \( \to \) slices and tubes.

---

Conductivity \( \sigma \) is not just a scalar multiplier!

For a closed surface: \( \oiint \mathbf{J} \cdot d\mathbf{S} = 0 \)

as steady current cannot be generated or absorbed inside a volume.

Equal amounts of current must enter and leave a closed surface \( \to \) *1st Kirchhoff's Law.*

![Flow of Steady Current Diagram](image-url)
Looking at small volume

For a small volume:

\[ \text{outflow} - \text{inflow} = \frac{\partial J_x}{\partial x} \delta x \delta y \delta z \]

Total net outflow = \[ \left[ \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right] \delta x \delta y \delta z \]

which goes down to zero as the volume shrinks. We can preserve the expression in brackets by the volume:

\[ \text{div} \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \]

The divergence (div) is the net outflow per unit volume for a very small volume.

For any \( S \) and \( v \):

\[ \iint \mathbf{J} \cdot d\mathbf{S} = 0 \]

\[ \therefore \text{div} \mathbf{J} = 0 \]
Physical meaning

- Steady current
  - No divergence (Laplacian field)

- Charge density increases
  - $\text{div} (-)$

- Charge density reduces
  - $\text{div} (+)$
Vector Analysis

- “Nabla” symbol
- Divergence
- Laplace Equation
Operators and “Nabla” Symbol

- “Del” is a vector differential operator
  - Describes the way in which some function changes
  - It is a convention for mathematical notation
  - Mathematically, it can be viewed as the derivative in multi-dimensional space

- Can be applied to various objects with different result!
  - makes many equations easier write and memorise

- Often “Del” is represented by the “nabla” symbol: \( \nabla \)

- \( \nabla \) was introduced in 1837 by the Irish mathematician and physicist Sir W.R. Hamilton

\[
\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}
\]
Gradient of Scalar Field

- A vector property of a scalar field
- Points in the direction of the greatest rate of increase of the scalar field

\[ \nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \]

\[ \text{grad} \ V \equiv \nabla V = \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z} \]
Indicates the source density
If the divergence is nonzero at some point then there must be a source or sink at that position

\[ \text{div } \mathbf{F} \equiv \nabla \cdot \mathbf{F} \]

\[ \nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \]

\[ \mathbf{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \]

\[ \text{div } \mathbf{F} \equiv \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \]
Divergence of a Vector Field

\[ \text{div } \mathbf{E} \triangleq \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{E} \cdot ds}{\Delta V}, \quad (3.95) \]

where \( S \) encloses the elemental volume \( \Delta V \). Instead of denoting the divergence of \( \mathbf{E} \) by \( \text{div } \mathbf{E} \), it is common practice to denote it as \( \nabla \cdot \mathbf{E} \). That is,

\[ \nabla \cdot \mathbf{E} = \text{div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \quad (3.96) \]

for a vector \( \mathbf{E} \) in Cartesian coordinates.

From the definition of the divergence of \( \mathbf{E} \) given by Eq. (3.95), field \( \mathbf{E} \) has positive divergence if the net flux out of surface \( S \) is positive, which may be “viewed” as if volume \( \Delta V \) contains a source of field lines. If the divergence is negative, \( \Delta V \) may be viewed as containing a sink of field lines because the net flux is into \( \Delta V \). For a uniform field \( \mathbf{E} \), the same amount of flux enters \( \Delta V \) as leaves it; hence, its divergence is zero and the field is said to be divergenceless.
Divergence Theorem

\[ \int_{V} \nabla \cdot \mathbf{E} \, dV = \oint_{S} \mathbf{E} \cdot d\mathbf{s} \quad (\text{divergence theorem}). \] (3.98)

Useful tool for converting integration over a volume to one over the surface enclosing that volume, and vice versa.
Example 3-11: Calculating the Divergence

Determine the divergence of each of the following vector fields and then evaluate them at the indicated points:

(a) \( \mathbf{E} = \hat{x}3x^2 + \hat{y}2z + \hat{z}x^2z \) at \((2, -2, 0)\):

Solution:

(a) \[ \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]
\[ = \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2z) \]
\[ = 6x + 0 + x^2 \]
\[ = x^2 + 6x. \]

At \((2, -2, 0)\), \( \nabla \cdot \mathbf{E} \bigg|_{(2,-2,0)} = 16. \)
one application of $\nabla$ gives three major derivatives:
- the gradient (scalar product)
- divergence (dot product)
- curl (cross product)

\[
\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}
\]

\[
\nabla V = \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z}
\]

\[
\text{div} \text{ grad } V \equiv \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}
\]

\[
\text{div} \ (\text{grad } V) \equiv \nabla \cdot (\nabla V) \equiv \nabla^2 V \equiv \Delta V
\]
Finally

Two equations: \( \mathbf{E} = -\text{grad} \, V \) and \( \text{div} \, \mathbf{J} = 0 \) plus Ohm’s Law \( \sigma \mathbf{E} = \mathbf{J} \)

\[ \therefore \text{div}(\sigma \text{grad} \, V) = 0 \]

Assume \( \sigma \) is independent of position:

\[ \text{div}(\text{grad} \, V) = 0 \]

or \( \nabla \cdot \nabla V = \nabla^2 V = 0 \) \( \text{Laplace’s equation} \)

in \( x, y, z: \)

\[ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \]

\[ \begin{align*}
\mathbf{E} &= -\text{grad} \, V \\
\text{div} \, \mathbf{J} &= 0 \\
\sigma \mathbf{E} &= \mathbf{J} \\
\text{or} \quad \nabla^2 V &= 0 \\
\text{Laplace’s equation} \\
\end{align*} \]

+ boundary conditions
- Potential
- Current density
An **electrical sensor** is a device capable of responding to an applied stimulus by generating an electrical signal whose voltage, current, or some other attribute is related to the intensity of the stimulus.

**Typical stimuli:** temperature, pressure, position, distance, motion, velocity, acceleration, concentration (of a gas or liquid), blood flow, etc.

**Sensing process** relies on measuring resistance, capacitance, inductance, induced electromotive force (emf), oscillation frequency or time delay, etc.

*Figure TF7-1:* Most cars use on the order of 100 sensors. (Courtesy Mercedes-Benz.)
Piezoresistivity

The Greek word *piezein* means to press

\[ R = R_0 \left(1 + \frac{\alpha F}{A_0}\right) \]

- \( R_0 \) = resistance when \( F = 0 \)
- \( F \) = applied force
- \( A_0 \) = cross-section when \( F = 0 \)
- \( \alpha \) = piezoresistive coefficient of material

*Figure TF7-2:* Piezoresistance varies with applied force.
Piezoresistors

Figure TF7-3: Piezoresistor films.

Figure TF7-4: Metal and silicon piezoresistors.
How to measure? Wheatstone Bridge

Wheatstone bridge is a high sensitivity circuit for measuring small changes in resistance.

\[ V_{out} = \frac{V_0}{4} \left( \frac{\Delta R}{R_0} \right) \]

**Figure TF7-5:** Wheatstone bridge circuit with piezoresistor.
Electrostatics and Steady Currents

- Electric fields
- Some Maths
- Steady Currents
- Electrostatics
Nomenclature - Electrostatics

- $E$ – electric field, $D$ – electric flux density,
- $V$ – voltage/potential, $q$ or $Q$ – electric charge
- $\rho$ - electric charge density, $\sigma$ - electric conductivity
- $v$ – volume, $s$ or $S$ – area, $l$ – length, $R$ – distance
- $F$ – force, $W$ – work or energy, $w$ – energy density
Coulomb’s Law

Electric field at point $P$ due to single charge

$$E = \hat{R} \frac{q}{4\pi \varepsilon R^2} \quad \text{(V/m)}$$

Electric force on a test charge placed at $P$

$$F = q'E \quad \text{(N)}$$

Electric flux density $\mathbf{D}$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\varepsilon = \varepsilon_r \varepsilon_0,$$

If $\varepsilon$ is independent of the magnitude of $\mathbf{E}$, then the material is said to be **linear** because $\mathbf{D}$ and $\mathbf{E}$ are related linearly, and if it is independent of the direction of $\mathbf{E}$, the material is said to be **isotropic**.

$$\varepsilon_0 = 8.85 \times 10^{-12} \simeq \frac{1}{36\pi} \times 10^{-9} \quad \text{(F/m)}$$
Figure 4-16: In the absence of an external electric field $E$, the center of the electron cloud is co-located with the center of the nucleus, but when a field is applied, the two centers are separated by a distance $d$.

Figure 4-17: A dielectric medium polarized by an external electric field $E$. 
Polarization Field

\[ D = \varepsilon_0 E + P \]

\[ P = \text{electric flux density induced by } E \]

\[ P = \varepsilon_0 \chi_e E, \quad (4.84) \]

where \( \chi_e \) is called the \textit{electric susceptibility} of the material. Inserting Eq. (4.84) into Eq. (4.83), we have

\[ D = \varepsilon_0 E + \varepsilon_0 \chi_e E \]

\[ = \varepsilon_0 (1 + \chi_e) E = \varepsilon E, \quad (4.85) \]
Electric Field Due to 2 Charges

with $R$, the distance between $q_1$ and $P$, replaced with $|\mathbf{R} - \mathbf{R}_1|$ and the unit vector $\hat{\mathbf{R}}$ replaced with $(\mathbf{R} - \mathbf{R}_1)/|\mathbf{R} - \mathbf{R}_1|$. Thus,

$$
E_1 = \frac{q_1(\mathbf{R} - \mathbf{R}_1)}{4\pi \varepsilon |\mathbf{R} - \mathbf{R}_1|^3} \quad \text{(V/m)}. \quad (4.17a)
$$

Similarly, the electric field at $P$ due to $q_2$ alone is

$$
E_2 = \frac{q_2(\mathbf{R} - \mathbf{R}_2)}{4\pi \varepsilon |\mathbf{R} - \mathbf{R}_2|^3} \quad \text{(V/m)}. \quad (4.17b)
$$

The electric field obeys the principle of linear superposition.

Hence, the total electric field $\mathbf{E}$ at $P$ due to $q_1$ and $q_2$ is

$$
\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2
= \frac{1}{4\pi \varepsilon} \left[ \frac{q_1(\mathbf{R} - \mathbf{R}_1)}{|\mathbf{R} - \mathbf{R}_1|^3} + \frac{q_2(\mathbf{R} - \mathbf{R}_2)}{|\mathbf{R} - \mathbf{R}_2|^3} \right]. \quad (4.18)
$$

**Figure 4-4:** The electric field $\mathbf{E}$ at $P$ due to two charges is equal to the vector sum of $\mathbf{E}_1$ and $\mathbf{E}_2$.

**Electric Field due to multiple charges**

$$
\mathbf{E} = \frac{1}{4\pi \varepsilon} \sum_{i=1}^{N} \frac{q_i(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3} \quad \text{(V/m)}. \quad (V/m).
$$
Electric Field Due to Charge Distributions

Field due to:

A differential amount of charge \( dq = \rho_v \, dV' \) contained in a differential volume \( dV' \) is

\[
dE = \hat{R}' \frac{dq}{4\pi \varepsilon R'^2} = \hat{R}' \frac{\rho_v \, dV'}{4\pi \varepsilon R'^2}, \quad (4.20)
\]

\[
E = \int_{V'} dE = \frac{1}{4\pi \varepsilon} \int_{V'} \hat{R}' \frac{\rho_v \, dV'}{R'^2}
\]

(volume distribution). \quad (4.21a)

\[
E = \frac{1}{4\pi \varepsilon} \int_{S'} \hat{R}' \frac{\rho_s \, ds'}{R'^2} \quad \text{(surface distribution)}, \quad (4.21b)
\]

\[
E = \frac{1}{4\pi \varepsilon} \int_{L'} \hat{R}' \frac{\rho_l \, dl'}{R'^2} \quad \text{(line distribution)}. \quad (4.21c)
\]
Forces, Fields and Energy

\[ F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r^2} \hat{r} \quad E = \frac{F}{Q_2} = \frac{Q_1}{4\pi \varepsilon_0 r^2} \hat{r} \]

A system of charges at rest is unstable and forces have to be applied to the charges to hold them in position. Work has to be done in assembling the charges:

\[ W = -\int_{\infty}^{r} F \cdot dr = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r} \]

“−” shows that work is done against the force \( F \).

The expression can be split into two statements:

\[ W = V_1 Q_2 \quad \text{and} \quad V_1 = -\int_{\infty}^{r} E \cdot dr = \frac{Q_1}{4\pi \varepsilon_0 r} \]

\( V_1 \) is the work done in bringing a unit charge to within a distance \( r \) of \( Q_1 \). This work is recovered when the charges are released → the system possesses potential energy and \( V \) is called the electrostatics potential.
Electric Potential Due to Charge Distributions

In electric circuits, we usually select a convenient node that we call ground and assign it zero reference voltage. In free space and material media, we choose infinity as reference with $V = 0$. Hence, at a point $P$

$$V = -\int_{\infty}^{P} E \cdot dl,$$  \hspace{1cm} (4.43)

For a point charge, $V$ at range $R$ is:

$$V = -\int_{\infty}^{R} \left( \hat{R} \frac{q}{4\pi \varepsilon R^2} \right) \cdot \hat{R} \, dR = \frac{q}{4\pi \varepsilon R}$$ \hspace{1cm} (V). \hspace{1cm} (4.45)

For continuous charge distributions:

$$V = \frac{1}{4\pi \varepsilon} \int_{V'} \frac{\rho_{v}}{R'} \, dV' \quad \text{(volume distribution)}, \hspace{1cm} (4.48a)$$

$$V = \frac{1}{4\pi \varepsilon} \int_{S'} \frac{\rho_{s}}{R'} \, ds' \quad \text{(surface distribution)}, \hspace{1cm} (4.48b)$$

$$V = \frac{1}{4\pi \varepsilon} \int_{L'} \frac{\rho_{l}}{R'} \, dl' \quad \text{(line distribution)}. \hspace{1cm} (4.48c)$$
No work has to be done in taking unit charge around closed loop. The system is **conservative**
Electric flux density

Define vector \( \mathbf{D} = \varepsilon_0 \mathbf{E} \) so that

\[
\mathbf{D} = \frac{Q_1}{4\pi r^2} \hat{r}
\]

If we take a surface integral over a sphere of radius \( r \) which has \( Q_1 \) at the centre:

\[
\iiint \mathbf{D} \cdot d\mathbf{s} = \iiint \frac{Q_1}{4\pi r^2} d\mathbf{s} = \frac{Q_1}{4\pi r^2} \iiint d\mathbf{s} = \frac{Q_1}{4\pi r^2} 4\pi r^2 = Q_1
\]

This result may be generalised by Gauss’s theorem → any surface, any charges inside:

\[
\iiint \mathbf{D} \cdot d\mathbf{s} = Q
\]

\[
\Psi = \iiint \mathbf{D} \cdot d\mathbf{s}
\]

**electric flux**

Hence \( \mathbf{D} \) is **electric flux density**

*Figure 4-8:* The integral form of Gauss’s law states that the outward flux of \( \mathbf{D} \) through a surface is proportional to the enclosed charge \( Q \). You can find proof in textbooks!
Gauss's Law in differential form

Also: \[ \iiint D \cdot ds = Q \]

For a distributed charge of density \( \rho \): \[ \iiint D \cdot ds = \iiint \rho \, dv \]

Divide by \( v \) surrounded by \( S \): \[ \frac{1}{v} \iiint D \cdot ds = \frac{1}{v} \iiint \rho \, dv \]

As \( v \to 0 \) the right hand side tends to \( \rho \): \[ \lim_{v \to 0} \frac{1}{v} \iiint D \cdot ds = \rho \]

\[ \text{Gauss's theorem: } \iiint D \cdot ds = \iiint \text{div } D \, dv \]
Gauss’s Law - summary

Application of the divergence theorem gives:

\[ \nabla \cdot \mathbf{D} = \rho_v \]

(Differential form of Gauss’s law),

\[ \int_{\mathcal{V}} \nabla \cdot \mathbf{D} \, d\mathcal{V} = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} = Q \]

Comparison of Eq. (4.27) with Eq. (4.28) leads to

\[ \oint_{S} \mathbf{D} \cdot d\mathbf{s} = Q \quad \text{(Integral form of Gauss’s law)} \]

The integral form of Gauss’s law is illustrated diagrammatically in Fig. 4-8; for each differential surface element \( d\mathbf{s} \), \( \mathbf{D} \cdot d\mathbf{s} \) is the electric field flux flowing outward of \( \mathcal{V} \) through \( d\mathbf{s} \), and the total flux through surface \( S \) equals the enclosed charge \( Q \). The surface \( S \) is called a Gaussian surface.

**Figure 4-8:** The integral form of Gauss’s law states that the outward flux of \( \mathbf{D} \) through a surface is proportional to the enclosed charge \( Q \).
Electrostatic Potential Energy

\[
W_e = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{\infty} \sum_{j>i} \frac{q_i q_j}{R_{ij}} = \frac{1}{8\pi\varepsilon} \sum_{i=1}^{\infty} \sum_{j\neq i} \frac{q_i q_j}{R_{ij}} = \frac{1}{2} \sum_{i=1}^{\infty} q_i V(R_i) = \frac{1}{2} \iiint dv(\rho V)
\]

Electrostatic potential energy density (Joules/volume)

\[
W_e = \frac{1}{2} \iiint dv(\rho V) = \frac{\varepsilon}{2} \iiint dv(\nabla \cdot \mathbf{E})V = \frac{\varepsilon}{2} \left[ -\iiint dv(\mathbf{E} \cdot (\nabla V)) + \iint V\mathbf{E} ds \right] = \frac{\varepsilon}{2} \iiint_{allspace} dv(\mathbf{E} \cdot \mathbf{E})
\]

Total electrostatic energy stored in a volume

\[
W_e = \frac{1}{2} \int_V \varepsilon E^2 \, dV \quad (J)
\]
Electrostatic Potential Energy

Electrostatic potential energy density (Joules/volume)

\[ w_e = \frac{W_e}{V} = \frac{1}{2} \varepsilon E^2 \quad \text{(J/m}^3\text{)} \]

Total electrostatic energy stored in a volume

\[ W_e = \frac{1}{2} \int_V \varepsilon E^2 \, dV \quad \text{(J)} \]
Electrostatic – Equations

\[ E = - \text{grad} V \]
\[ \text{div} \mathbf{D} = \rho \]
\[ \mathbf{D} = \varepsilon_0 \mathbf{E} \]

A system of first order differential equations

Assuming \( \varepsilon = 1 \! \):

\[ \text{div} \mathbf{D} = \text{div}(\varepsilon_0 \mathbf{E}) = \text{div}(-\varepsilon_0 \text{grad} V) = \rho \]

\[ \nabla \cdot \nabla V = \nabla^2 V = -\frac{\rho}{\varepsilon_0} \]

Poisson’s equation

in 3D:
\[ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon_0} \]

in 2D:
\[ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\rho}{\varepsilon_0} \]

Compare with steady current equations!
Poisson’s & Laplace’s Equations

With \( \mathbf{D} = \varepsilon \mathbf{E} \), the differential form of Gauss’s law given by Eq. (4.26) may be cast as

\[
\nabla \cdot \mathbf{E} = \frac{\rho_y}{\varepsilon} . \tag{4.57}
\]

Inserting Eq. (4.51) in Eq. (4.57) gives

\[
\nabla \cdot (\nabla V) = -\frac{\rho_y}{\varepsilon} . \tag{4.58}
\]

Given Eq. (3.110) for the Laplacian of a scalar function \( V \),

\[
\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} , \tag{4.59}
\]

Eq. (4.58) can be cast in the abbreviated form

\[
\nabla^2 V = -\frac{\rho_y}{\varepsilon} \quad \text{(Poisson’s equation).} \tag{4.60}
\]

This is known as Poisson’s equation. For a volume \( V’ \) containing a volume charge density distribution \( \rho_y \), the solution for \( V \) derived previously and expressed by Eq. (4.48a) as

\[
V = \frac{1}{4\pi \varepsilon} \int_{V'} \frac{\rho_y}{R'} dV' \tag{4.61}
\]
Boundary Conditions

\[ \rho_s \]

\[ \oint E \cdot dl = 0, \]

\[ E_{\text{above}} = E_{\text{below}}, \]
Boundary Conditions

\[ \oint_{\text{surface}} \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}} = \rho_s S \]

\[ \left( D_{\text{above}}^\perp - D_{\text{below}}^\perp \right) S = \rho_s S \]

For \( \varepsilon = 1 \)

\[ E_{\text{above}}^\perp - E_{\text{below}}^\perp = \frac{1}{\varepsilon} \rho_s \]
Even with surface charges $E$ is finite, so when $dl$ goes to zero,

$$V_{\text{above}} - V_{\text{below}} = -\int_{a}^{b} E \cdot dl;$$

and

$$V_{\text{above}} = V_{\text{below}}.$$
**Boundary Conditions**

![Diagram of a boundary between two dielectric media](image)

**Figure 4-18:** Interface between two dielectric media.

1. **Electric Fields**
   \[
   E_{1t} = E_{2t} \quad (\text{V/m}). \quad (4.90)
   \]

2. **Permittivity and Dielectric Vectors**
   \[
   \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}. \quad (4.91)
   \]

3. **Normal Component of Charge Density**
   \[
   \hat{n}_2 \cdot (D_1 - D_2) = \rho_s \quad (\text{C/m}^2). \quad (4.94)
   \]

4. **Normal Component of Electric Displacement**
   \[
   D_{1n} - D_{2n} = \rho_s \quad (\text{C/m}^2). \quad (4.94)
   \]

**The normal component of D changes abruptly at a charged boundary between two different media in an amount equal to the surface charge density.**
Fundamentals of Electrostatics

\[ V = \frac{1}{4\pi \varepsilon_0} \int \rho \, dV \]

\[ \nabla^2 V = -\rho \varepsilon_0 \]

\[ \nabla \cdot E = \rho \varepsilon_0 \quad \nabla \times E = 0 \]

\[ E = -\nabla V \]

\[ V = -\int E \cdot dl \]
## Summary of Boundary Conditions

### Table 4-3: Boundary conditions for the electric fields.

<table>
<thead>
<tr>
<th>Field Component</th>
<th>Any Two Media</th>
<th>Medium 1 Dielectric $\varepsilon_1$</th>
<th>Medium 2 Conductor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangential $E$</td>
<td>$E_{1t} = E_{2t}$</td>
<td>$E_{1t} = E_{2t} = 0$</td>
<td></td>
</tr>
<tr>
<td>Tangential $D$</td>
<td>$D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$</td>
<td>$D_{1t} = D_{2t} = 0$</td>
<td></td>
</tr>
<tr>
<td>Normal $E$</td>
<td>$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$</td>
<td>$E_{1n} = \rho_s/\varepsilon_1$</td>
<td>$E_{2n} = 0$</td>
</tr>
<tr>
<td>Normal $D$</td>
<td>$D_{1n} - D_{2n} = \rho_s$</td>
<td>$D_{1n} = \rho_s$</td>
<td>$D_{2n} = 0$</td>
</tr>
</tbody>
</table>

Notes: (1) $\rho_s$ is the surface charge density at the boundary; (2) normal components of $E_1$, $D_1$, $E_2$, and $D_2$ are along $\hat{n}_2$, the outward normal unit vector of medium 2.

---

**Electrostatics:** $E = 0$ in any “reasonable” conductor
Even when current flows, remember $E \sim 0$ in a good conductor
Conductors

Electrostatics:
Net electric field inside a conductor is zero

Figure 4-20: When a conducting slab is placed in an external electric field $E_1$, charges that accumulate on the conductor surfaces induce an internal electric field $E_i = -E_1$. Consequently, the total field inside the conductor is zero.
Electrostatics: At conductor boundary, \( \mathbf{E} \) field direction is always perpendicular to conductor surface.
Voids in Conductors

1. A charged particle (+q) inside a conductor.
2. A conductor with an electric field (E ≠ 0) inside and an electric field of zero (E = 0) outside.
3. A Gaussian surface with charge q inside a conductor.
Forces on Conductors

\[ \mathbf{E} = \mathbf{E}_{\text{patch}} + \mathbf{E}_{\text{other}}. \]

\[ E_{\text{above}} = E_{\text{below}}, E_{\text{below}} = 0 \]

\[ D_{\text{above}} = D_{\text{below}} + \rho_s, D_{\text{below}} = 0 \]

\[ D_{\text{above}} / \varepsilon = \frac{\rho_s}{\varepsilon} = E_{\text{above}} = E_{\text{ext}} + E_\rho, E_\rho = \frac{\rho_s}{2\varepsilon} \Rightarrow E_{\text{ext}} = \frac{\rho_s}{2\varepsilon} \]

\[ f_{\text{above}} = \rho_s E_{\text{ext}} = \frac{\rho_s^2}{2\varepsilon} = \frac{\varepsilon (E_{\text{above}})}{2} \]
Electrostatic Screening
Electrostatic Screening

Surface: Electric potential (V)  Contour: Electric potential (V)
Electrostatic Screening
Capacitance

When a conductor has excess charge, it distributes the charge on its surface in such a manner as to maintain a zero electric field everywhere within the conductor, thereby ensuring that the electric potential is the same at every point in the conductor.

The *capacitance* of a two-conductor configuration is defined as

\[ C = \frac{Q}{V} \quad \text{(C/V or F)}, \quad (4.105) \]

*Figure 4-23:* A dc voltage source connected to a capacitor composed of two conducting bodies.
Capacitance

For any two-conductor configuration:

\[ C = \frac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{\int_l \mathbf{E} \cdot d\mathbf{l}} \quad (\text{F}), \]

For any resistor:

\[ R = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} \quad (\Omega). \quad (4.110) \]

For a medium with uniform \( \sigma \) and \( \varepsilon \), the product of Eqs. (4.109) and (4.110) gives

\[ RC = \frac{\varepsilon}{\sigma}. \quad (4.111) \]

This simple relation allows us to find \( R \) if \( C \) is known, or vice versa.

Figure 4-23: A dc voltage source connected to a capacitor composed of two conducting bodies.

We can use Tubes and Slices! Tubes of D, Slices of V.
Dielectric Materials

Bound charges and free charges:

\[ \rho = \rho_b + \rho_f \]

Let’s introduce polarisation:

\[ \rho_b = -\nabla \cdot \mathbf{P} \]

Microscopic Gauss’s law:

\[ \varepsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f \]

Macroscopic Gauss’s law:

\[ \nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f \quad \Rightarrow \quad \mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \Rightarrow \quad \nabla \cdot \mathbf{D} = \rho_f \]

*Figure 4-17*: A dielectric medium polarized by an external electric field \( \mathbf{E} \).
Plane Capacitor

\[ E = \frac{Q - q}{\varepsilon_0 S} \]

\[ V = \int -E \, dl = \frac{Q - q}{\varepsilon_0 S} \, l. \]

\[ \frac{Q}{Q - q} = 1 + \frac{q}{Q - q} = 1 + \chi_e = \varepsilon_r, \]

\[ C = \frac{Q}{V} = \frac{Q}{Q - q} \frac{\varepsilon_0 S}{l}. \]

\[ C = \frac{\varepsilon_0 \varepsilon_r S}{l} = \frac{\varepsilon S}{l}. \]
Forces on Dielectric boundaries

\[ f_{\text{above}}^\perp = \frac{(D^\perp)^2}{2} \left( \frac{1}{\varepsilon_{\text{above}}} - \frac{1}{\varepsilon_{\text{below}}} \right) + \frac{(E^\parallel)^2}{2} \left( \varepsilon_{\text{below}} - \varepsilon_{\text{above}} \right) \]
Energy in Capacitor

Generally, the electrostatic energy may be expressed as:

\[ W = \frac{1}{2} QV = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2 \]

C is constant. \( dq = C \, dV \), so \( dW = CV \, dV \). Integration over charging process from \( q=0 \) to \( q=Q \) gives the energy.
Forces on Dielectric boundaries

Pull out dielectric

\[ dW = F_{me} \, dx \]

Electric force

\[ F_{me} = -F \]

\[ F = -\frac{dW}{dx} \]

\[ W = \frac{1}{2} CV^2 \quad \text{or} \quad W = \frac{1}{2} \frac{Q^2}{C} \]

\[ C = \frac{\epsilon_0 w}{d} (\epsilon_r l - \chi_e x) \]

Dielectric susceptibility \( \chi_e \).

\[ \epsilon \equiv \epsilon_0 (1 + \chi_e) \]
Forces on Dielectric boundaries

**Fixed charge**

\[
W = \frac{1}{2} \frac{Q^2}{C}
\]

\[
F = -\frac{dW}{dx} = \frac{1}{2} \frac{Q^2}{C^2} \frac{dC}{dx} = \frac{1}{2} V^2 \frac{dC}{dx}.
\]

\[
\frac{dC}{dx} = -\frac{\varepsilon_0 \chi_e w}{d},
\]

\[
F = -\frac{\varepsilon_0 \chi_e w}{2d} V^2.
\]

\[
C = \frac{\varepsilon_0 w}{d} (\varepsilon_r l - \chi_e x)
\]
Forces on Dielectric boundaries

Fixed voltage

\[ W = \frac{1}{2} C V^2 \]

Battery also does the work!

\[ dW = F_{me} \, dx + V \, dQ, \]

\[ F = -\frac{dW}{dx} + V \frac{dQ}{dx} = -\frac{1}{2} V^2 \frac{dC}{dx} + V^2 \frac{dC}{dx} = \frac{1}{2} V^2 \frac{dC}{dx}, \]

\[ \frac{dC}{dx} = -\frac{\varepsilon_0 \chi_e w}{d}, \quad F = -\frac{\varepsilon_0 \chi_e w}{2d} \, V^2. \]

Same answer!
Breakdown Voltage

\[ V = - \int_{0}^{d} \mathbf{E} \cdot d\mathbf{l} = - \int_{0}^{d} (\hat{\mathbf{z}} E) \cdot \hat{\mathbf{z}} \, dz = Ed, \quad (4.112) \]

and the capacitance is

\[ C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d}, \quad (4.113) \]

where use was made of the relation \( E = Q/\varepsilon A \).

From \( V = Ed \), as given by Eq. (4.112), \( V = V_{\text{br}} \) when \( E = E_{ds} \), the dielectric strength of the material. According to Table 4-2, \( E_{ds} = 30 \text{ (MV/m)} \) for quartz. Hence, the breakdown voltage is

\[ V_{\text{br}} = E_{ds}d = 30 \times 10^6 \times 10^{-2} = 3 \times 10^5 \text{ V}. \]
The dielectric strength $E_{ds}$ is the largest magnitude of $E$ that the material can sustain without breakdown.

**Table 4-2:** Relative permittivity (dielectric constant) and dielectric strength of common materials.

<table>
<thead>
<tr>
<th>Material</th>
<th>Relative Permittivity, $\varepsilon_r$</th>
<th>Dielectric Strength, $E_{ds}$ (MV/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air (at sea level)</td>
<td>1.0006</td>
<td>3</td>
</tr>
<tr>
<td>Petroleum oil</td>
<td>2.1</td>
<td>12</td>
</tr>
<tr>
<td>Polystyrene</td>
<td>2.6</td>
<td>20</td>
</tr>
<tr>
<td>Glass</td>
<td>4.5–10</td>
<td>25–40</td>
</tr>
<tr>
<td>Quartz</td>
<td>3.8–5</td>
<td>30</td>
</tr>
<tr>
<td>Bakelite</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>Mica</td>
<td>5.4–6</td>
<td>200</td>
</tr>
</tbody>
</table>

$\varepsilon = \varepsilon_r \varepsilon_0$ and $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.
For a traditional parallel-plate capacitor, what is the maximum attainable energy density?

Mica has one of the highest dielectric strengths \( \sim 2 \times 10^8 \text{ V/m} \). If we select a voltage rating of 1 V and a breakdown voltage of 2 V (50% safety), this will require that \( d \) be no smaller than 10 nm. For mica, \( \varepsilon = 6\varepsilon_0 \) and \( \rho = 3 \times 10^3 \text{ kg/m}^3 \).

Hence:

\[
W' = 90 \text{ J/kg} = 2.5 \times 10^{-2} \text{ Wh/kg}.
\]

By comparison, a lithium-ion battery has \( W' = 1.5 \times 10^2 \text{ Wh/kg} \), almost 4 orders of magnitude greater.
Fluid Gauge

The two metal electrodes in Fig. TF9-1(a), usually rods or plates, form a capacitor whose capacitance is directly proportional to the permittivity of the material between them. If the fluid section is of height $h_f$ and the height of the empty space above it is $(h - h_f)$, then the overall capacitance is equivalent to two capacitors in parallel, or

$$C = C_f + C_a = \varepsilon_f w \frac{h_f}{d} + \varepsilon_a w \frac{(h - h_f)}{d},$$

where $w$ is the electrode plate width, $d$ is the spacing between electrodes, and $\varepsilon_f$ and $\varepsilon_a$ are the permittivities of the fluid and air, respectively. Rearranging the expression as a linear equation yields

$$C = kh_f + C_0,$$
Figure TF9-2: Interdigital capacitor used as a humidity sensor.
Figure TF9-3: Pressure sensor responds to deflection of metallic membrane.
Planar capacitors

**Figure TF9-4:** Concentric-plate capacitor.

**Figure TF9-5:** (a) Adjacent-plates capacitor; (b) perturbation field.
Fingerprint Imager

Figure TF9-6: Elements of a fingerprint matching system. (Courtesy of IEEE Spectrum.)

Figure TF9-7: Fingerprint representation. (Courtesy of Dr. M. Tartagni, University of Bologna, Italy.)
Useful Formulas

<table>
<thead>
<tr>
<th>Name</th>
<th>Differential Form</th>
<th>Integral Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’s law</td>
<td>$\nabla \cdot \mathbf{D} = \rho_v$</td>
<td>$\int_S \mathbf{D} \cdot d\mathbf{s} = Q$</td>
</tr>
<tr>
<td>Kirchhoff’s law</td>
<td>$\nabla \times \mathbf{E} = 0$</td>
<td>$\int_C \mathbf{E} \cdot d\mathbf{l} = 0$</td>
</tr>
</tbody>
</table>

Electric Field

- **Point charge**
  \[ \mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi \varepsilon R^2} \]

- **Many point charges**
  \[ \mathbf{E} = \frac{1}{4\pi \varepsilon} \sum_{i=1}^{N} \frac{q_i (\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3} \]

- **Volume distribution**
  \[ \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{V'} \hat{\mathbf{R}}' \frac{\rho_v \, dV'}{R'^2} \]

- **Surface distribution**
  \[ \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2} \]

- **Line distribution**
  \[ \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_l \, dl'}{R'^2} \]

- **Infinite sheet of charge**
  \[ \mathbf{E} = \hat{z} \frac{\rho_s}{2\varepsilon_0} \]

- **Infinite line of charge**
  \[ \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{r} \frac{D_r}{\varepsilon_0} = \hat{r} \frac{\rho_l}{2\pi \varepsilon_0 r} \]

- **Dipole**
  \[ \mathbf{E} = \frac{q d}{4\pi \varepsilon_0 R^3} (\hat{\mathbf{R}} \ 2 \cos \theta + \hat{\theta} \sin \theta) \]

- **Relation to $V$**
  \[ \mathbf{E} = -\nabla V \]