The pulse transfer function
Motivation

Time-varying nature and intersample behaviour are important: *hard* math ("lifting") can address these.

**Stroboscopic** point of view: assuming perfect synchronization of sampling at every step and at every point in the controlled system, variables are considered only at the sampling instants.

Leads to easy-to-devise- and easy-to-use models: a controller’s point of view, a discrete-time one.
A computer’s view of digital control

If A-D and D-A are perfectly synchronized, and sampling is ideal, controller could be designed by lumping what is out of it in an “equivalent” discrete-time system.
The pulse transfer function (ZOH case)

\[ H(z) \] transforms discrete- to discrete
The pulse transfer function (ZOH case)

$H(z)$ transforms discrete- to discrete

If plant has transfer function (continuous) $G(s)$, what is the pulse transfer function $H(z)$?
Ogata’s Ch. 3 concerns a “pulse transfer function", which is however **different** from ours.

Same term, but **different** block diagrams-
compare Ogata’s Fig. 3.20 with that on slide 5.
Calculation of the pulse transfer function

We use a “recipe” based on the observation:

*The pulse transfer function is uniquely determined by the response to a given signal (easiest, the step-response)*

1. Determine the (continuous-time) step-response $y(\cdot)$ of $G(s)$;
2. Sample $y(\cdot) \sim \{y(kT)\}_{k=0,...}$, and $\mathcal{Z}$-transform $\{y(kT)\}_{k=0,...}$;
3. Divide result by the $\mathcal{Z}$-transform of the (DT) step function, i.e. $\frac{z}{z-1}$. 
Example

Consider $G(s) = \frac{1}{s+2}$. Assume this block is preceded by a ZOH-block and followed by an ideal sampler; find the pulse transfer function.
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- \( \mathcal{L} \)-transform of step response in continuous-time:

\[
\frac{1}{s+2} - \frac{1}{s+2} = -\frac{1}{2} + \frac{1}{2} \sim -\frac{1}{2} e^{-2t} + \frac{1}{2} \delta(t)
\]
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- \( \mathcal{L} \)-transform of step response in continuous-time:

\[
\frac{1}{s+2} \cdot \frac{1}{s} = \frac{-1}{2} \cdot \frac{1}{s+2} + \frac{1}{2} \cdot \frac{1}{s} \sim -\frac{1}{2}e^{-2t} + \frac{1}{2}1(t)
\]

- Sample with period \( T \): \(-\frac{1}{2}e^{-2kT} + \frac{1}{2}\)
Example

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- $\mathcal{L}$-transform of step response in continuous-time:

  \[
  \frac{1}{s + 2} \frac{1}{s} = \frac{-\frac{1}{2}}{s + 2} + \frac{1}{2} \frac{s}{s} \rightsquigarrow -\frac{1}{2} e^{-2t} + \frac{1}{2} 1(t)
  \]

- Sample with period $T$: $-\frac{1}{2} e^{-2kT} + \frac{1}{2}$

- $Z$-transform:

  \[
  -\frac{1}{2} \frac{z}{z - e^{-2T}} + \frac{1}{2} \frac{z}{z - 1} = \frac{z(e^{2T} - 1)}{2(z - 1)(e^{2T}z - 1)}
  \]
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  \]

- Sample with period \( T \):
  \[
  -\frac{1}{2} e^{-2kT} + \frac{1}{2}
  \]

- \( \mathcal{Z} \)-transform:
  \[
  -\frac{1}{2} \frac{z}{z-e^{-2T}} + \frac{1}{2} \frac{z}{z-1} = \frac{z(e^{2T} - 1)}{2(z-1)(e^{2T}z-1)}
  \]

- Divide by \( \frac{z}{z-1} \):
  \[
  \frac{(e^{2T} - 1)}{2(e^{2T}z - 1)}
  \]
Exercise

Let $a, b, c \in \mathbb{R}$, $G(s) = \frac{as}{(s+b)(s+c)}$; denote the sampling period with $T$. Assume ZOH before, and ideal sampling after. Find the pulse transfer function.
Exercise

Let \( a, b, c \in \mathbb{R}, \ G(s) = \frac{as}{(s+b)(s+c)}; \) denote the sampling period with \( T. \) Assume ZOH before, and ideal sampling after. Find the pulse transfer function.

Assume that \( c \neq b, \) write \( G(s) \frac{1}{s} = \frac{a}{(s+b)(s+c)} = \frac{\frac{a}{s+c}}{s+c} + \frac{-\frac{a}{s+b}}{s+b}, \) so

\[
\mathcal{L}^{-1} \left( G(s) \frac{1}{s} \right) = \frac{a}{b-c} e^{-ct} - \frac{a}{b-c} e^{-bt}
\]

Sampled signal is \( \left\{ \frac{a}{b-c} \left( e^{-ckT} - e^{-bkT} \right) \right\}_{k=0,...} \) with \( Z \)-transform

\[
\frac{a}{b-c} \left( \frac{z}{z - e^{-cT}} - \frac{z}{z - e^{-bT}} \right)
\]

The pulse transfer function is

\[
\frac{a}{b-c} \frac{z-1}{z} \left( \frac{z}{z - e^{-cT}} - \frac{z}{z - e^{-bT}} \right) = \frac{a}{b-c} \frac{(z - 1) \left( e^{-cT} - e^{-bT} \right)}{(z - e^{-cT}) (z - e^{-bT})}
\]
Since the ZOH-PTF is uniquely determined by the response to a given signal, why not using the impulse?

Laplace-transform of impulse response:

\[ G(s) \mathcal{L}(\overline{1}(\cdot) - \overline{1}(\cdot - \tau)) = G(s) \left( \frac{1 - e^{-sT}}{s} \right) \]

Now need to invert this, sample, and \( \mathcal{Z} \)-transform...

...easier to work with \( \overline{1}(\cdot) \), which is a “unity” for the ZOH (i.e. is unaltered by ZOH).
Remarks

- Additional zeroes may be present in the pulse transfer function.
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- Complicated for realistic transfer functions, but Matlab gives a helping hand.
The \texttt{c2d} function of Matlab

Previous example for $a = 1$, $b = 2$, $c = 3$, $T = 0.01$:

\begin{verbatim}
contsys=tf([1 0],[1 5 6]);
discrsys=c2d(contsys,0.01,'zoh');
impulse(contsys);
figure;
impulse(discrsys);
discrsys
Transfer function:
0.009753 z - 0.009753
---------------------
z^2 - 1.951 z + 0.9512
Sampling time: 0.01
\end{verbatim}

Note \textit{scaling} of the discrete-time case due to ZOH!
Example: pulse transfer function, sampling, \textit{Matlab-1}

Consider \( G(s) = \frac{2}{s^2+2s+2} \) preceded by ZOH and followed by sampler.

Step response in \( \mathcal{L} \)-domain:

\[
\frac{2}{s^2 + 2s + 2} - \frac{1}{s} = \frac{-s - 2}{s^2 + 2s + 2} + \frac{1}{s} \sim 1 - e^{-t} \cos(t) - e^{-t} \sin(t)
\]
Example: pulse transfer function, sampling, Matlab-1

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\]

Sample at \( T \):

\[
1 - e^{-kT} \cos(kT) - e^{-kT} \sin(kT)
\]

\[
\sim z \left( e^{2T} z - e^T (z - 1) \sin(T) - e^T (z + 1) \cos(T) + 1 \right) / (z - 1) (e^{2T} z^2 - 2e^T z \cos(T) + 1)
\]
Example: pulse transfer function, sampling, \texttt{Matlab-2}

\[
\frac{2}{s^2+2s+2} \sim \frac{z(e^{2T}z - e^T(z-1)\sin(T) - e^T(z+1)\cos(T)+1)}{(z-1)(e^{2T}z^2 - 2e^Tz\cos(T)+1)} =: H(z, T)
\]
Example: pulse transfer function, sampling, Matlab-2

\[
\frac{2}{s^2 + 2s + 2} \sim \frac{z(e^{2T}z - e^T(z-1)\sin(T) - e^T(z+1)\cos(T) + 1)}{(z-1)(e^{2T}z^2 - 2e^Tz\cos(T) + 1)} =: H(z, T)
\]

Bode plot of original TF:

Bandwidth is approximately 10 rad/s. Nyquist says sample at least at 20 rad/s \( \sim T = \frac{\pi}{20} \, s \).
Example: pulse transfer function, sampling, \texttt{Matlab-2}

\[
\frac{2}{s^2+2s+2} \sim \frac{z(e^{2T}z-e^T(z-1)\sin(T)-e^T(z+1)\cos(T)+1)}{(z-1)(e^{2T}z^2-2e^Tz\cos(T)+1)} =: H(z, T)
\]

Sampling at \( T = \frac{\pi}{20} \) s yields

\[
H(z, T) = \frac{0.0221915(1.\cdot z+0.900491)}{1.\cdot z^2-1.68823z+0.730403}.
\]

Impulse response is (blue=original):
Example: pulse transfer function, sampling, \texttt{Matlab-2}

\[
\frac{2}{s^2+2s+2} \sim \frac{z(e^{2T}z-e^T(z-1)\sin(T)-e^T(z+1)\cos(T)+1)}{(z-1)(e^{2T}z^2-2e^Tz\cos(T)+1)} =: H(z, T)
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Sampling at \( T = \frac{\pi}{20} \) \( s \) yields

\[
H(z, T) = \frac{0.0221915(1.0z+0.900491)}{1.0z^2-1.68823z+0.730403}.
\]

Response to \( \sin(t) \) is (blue=original):

![Graph showing the response to \( \sin(t) \)]
Example: pulse transfer function, sampling, \texttt{Matlab-2}

\[
\frac{2}{s^2+2s+2} \sim \frac{z\left(e^{2T}z - e^T(z-1)\sin(T) - e^T(z+1)\cos(T) + 1\right)}{(z-1)\left(e^{2T}z^2 - 2e^Tz \cos(T) + 1\right)} \quad \Rightarrow \quad H(z, T)
\]

Sampling at \( T = 3.3\text{s} \) yields \( H(z, T) = \frac{1.042z + 0.0296}{z^2 + 0.070z + 0.0013} \).

Impulse response is (blue=original):
Example: pulse transfer function, sampling, \texttt{Matlab}-2

\[
\frac{2}{s^2+2s+2} \sim \frac{z\left(e^{2T}z-e^T(z-1)\sin(T)-e^T(z+1)\cos(T)+1\right)}{(z-1)\left(e^{2T}z^2-2e^Tz\cos(T)+1\right)} =: H(z, T)
\]

Sampling at \(T = 3.3s\) yields \(H(z, T) = \frac{1.042z+0.0296}{z^2+0.070z+0.0013}\).

Response to \(\sin(t)\) is is (blue=original):

![Graph showing the response to \(\sin(t)\)]
Example: pulse transfer function, sampling, Matlab-3

```matlab
bode(sysc)
grid on
figure
impulse(sysc)
Ts=pi/20
sysd=c2d(sysc,Ts)
hold on
impulse(sysd)
figure
bode(sysd)
t=0:0.001:50
u=sin(t);
yc=lsim(sysc,u,t);
figure
plot(t,yc)
t=0:Ts:50;
u=sin(t);
yd=lsim(sysd,u,t);
figure
plot(t,yd,'r')
hold on
plot(t,yc,'b')
```

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A dynamical interpretation of zeroes

A zero of $H(z)$ at $\alpha$ corresponds to a signal $\{\alpha^k\}_{k=0,...}$ which is “blocked” by the system

$$H(z) = \frac{(z - \alpha)n(z)}{d(z)}$$

$\mathcal{Z}$-transform of response to $\{\alpha^k\}_{k=0,...}$ (zero i.c.) is

$$\frac{(z - \alpha)n(z)}{d(z)} \frac{z}{z - \alpha} = \frac{zn(z)}{d(z)}$$

Partial fraction expansion $\sim$ blocking of $\{\alpha^k\}_{k=0,...}$.

Another feature to reckon with when dealing with digital control systems...
Mapping of poles and zeroes

\[ s \rightarrow z = e^{sT} \]
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Poles \( p_i \) of \( G(s) \) mapped to poles \( z_i := e^{p_i T} \) of \( G(z) \)

\( ? \) ...and the zeroes \( ? \)
Mapping of poles and zeroes

\[ s \rightarrow z = e^{sT} \]

No simple formula for zeroes.
Generally, if \( n \) poles are present in \( G(s) \),
then there are \( n - 1 \) zeroes present in \( G(z) \).
Even if there were less than \( n - 1 \) zeroes in \( G(s) \)!
Important remark

Note that if the pole $\lambda = a + jb$ is in

$$\mathbb{C}_- := \{z \in \mathbb{C} \mid \text{real part of } z \text{ is } < 0\}$$

(left half-plane), then it gets mapped to

$$e^{\lambda T} = e^{aT} e^{jbT}$$

whose magnitude is

$$|e^{\lambda T}| = |e^{aT}| \cdot 1 = \frac{1}{e^{-aT}} < 1$$

Stable/unstable continuous poles
are mapped into stable/unstable discrete poles
Pulse transfer function for cascaded elements

Two different situations $\rightsquigarrow$ different PTFs. (Note: I assume sampling is synchronized and $T$ is the same!)
Pulse transfer function for cascaded elements

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E.g. take $G(s) = \frac{1}{s}$, $H(s) = \frac{1}{s+1}$. 
Pulse transfer function for cascaded elements

Two different situations $\leadsto$ different PTFs. (Note: I assume sampling is synchronized and $T$ is the same!)

E.g. take $G(s) = \frac{1}{s}$, $H(s) = \frac{1}{s+1}$.

Upper situation leads to $G(z) \cdot H(z) = \frac{(e^T - 1)T}{(z-1)(ze^T - 1)}$. 
Pulse transfer function for cascaded elements

Two different situations $\Rightarrow$ different PTFs. (Note: I assume sampling is synchronized and $T$ is the same!)

E.g. take $G(s) = \frac{1}{s}$, $H(s) = \frac{1}{s+1}$.

Upper situation leads to $G(z) \cdot H(z) = \frac{(e^T-1)T}{(z-1)(ze^T-1)}$.

Lower situation leads to

$$(G \cdot H)(z) = \frac{-1 - T + z + e^T + ze^T(T-1)}{(z - 1)(ze^T - 1)}$$
Remarks

- If sampling not synchronized or at different rates, situation more complicated ("multirate systems");
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- Sampler between $G(s)$ and $H(s)$ makes the difference;
Remarks

• If sampling not synchronized or at different rates, situation more complicated ("multirate systems");

• Sampler between $G(s)$ and $H(s)$ makes the difference;

• For closed-loop systems, situation is analogous, see e.g. Ogata pp. 110–113.
From real system to mathematical model

\[ G(z) \] is pulse transfer function of plant.

All discrete signals now.
From real system to mathematical model

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From real system to mathematical model

\[ G(z) \] is pulse transfer function of plant.
All discrete signals now.
Where now, and where to?

- Good understanding of sampling issues;
Where now, and where to?

• Good understanding of sampling issues;

• From $\mathcal{L}$-domain model of continuous-time plant (transfer function) to $z$-domain discrete-time model (pulse transfer function), no problem;
Where now, and where to?

- Good understanding of sampling issues;
- From $\mathcal{L}$-domain model of continuous-time plant (transfer function) to $z$-domain discrete-time model (pulse transfer function), no problem;
- However, models do not come only in transfer function format!
The state representation, and discretization of continuous state-space systems
Warning

In this part of the course we make **strenuous** use of Linear Algebra concepts and tools.

If you are uncomfortable with eigenvalues, eigenvectors, determinants, characteristic polynomials, and all that stuff, you **must** revise these notions in order to follow proficiently this part of the course.

**Knowledge is cumulative.**
Want to transform continuous-time state-space model of plant plus D-A and A-D conversion in discrete-time state-space model.
State: the basic idea

To follow from now on, just look at the chessboard
State: the basic idea

To follow from now on, just look at the chessboard

- The state contains all the relevant information about the future behavior of the system
- The state is the memory of the system
- Independence of past and future given the state
State: two points of view

In the rest of this course, two points of view adopted on state and state representations:

1) The state is *given* as a “natural” set of attributes splitting past and future of the system
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2) The state is *constructed* from the equations/transfer function of the system, in order to exploit the remarkable properties and insight offered by state-space equations
State: two points of view

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Contradictory point of views, in some sense, but each with its merits.
Discrete-time state-space systems

\[
x(k + 1) = Ax(k) + Bu(k) \\
y(k) = Cx(k) + Du(k)
\]

\(x\) state, \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}\)
Discrete-time state-space systems

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Given \(x(0)\), the state at time \(k \geq 1\) is

\[
x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j)
\]
Discrete-time state-space systems

\[ x(k + 1) = Ax(k) + Bu(k) \]
\[ y(k) = Cx(k) + Du(k) \]

\( x \) state, \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m} \)

Given \( x(0) \), the state at time \( k \geq 1 \) is

\[ x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j) \]

Easy to conclude that

\[ y(k) = CA^k x(0) + \sum_{j=0}^{k-1} CA^{k-1-j} Bu(j) + Du(k) \]
Example: evolution of a population

A population is divided in three age groups: young ($y(\cdot)$), mature ($m(\cdot)$) and old ($v(\cdot)$); the dynamics are:

$$x(k + 1) = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.9 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix} x(k)$$

where $x(k) = \begin{bmatrix} y(k) \\ m(k) \\ v(k) \end{bmatrix}$. Evolution is

$$x(k) = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.9 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix}^k x(0)$$

¿How to compute efficiently the powers of $A$?
Solving the state equation efficiently
Recall definition of eigenvalue and eigenvector:

\[ Av = \lambda v_\lambda \]

where eigenvalue \( \lambda \in \mathbb{C} \), eigenvector \( v_\lambda \in \mathbb{C}^n \). The \( \lambda \)s are the roots of the characteristic polynomial:

\[ \chi_A(z) := \det(zI - A) \]
Solving the state equation efficiently

If $A$ has all distinct $\lambda$s (our case for this course) then the eigenvectors are linearly independent. Follows

$$A[\begin{bmatrix} v_{\lambda_1} & v_{\lambda_2} & \cdots & v_{\lambda_n} \end{bmatrix}] = [\begin{bmatrix} v_{\lambda_1} & v_{\lambda_2} & \cdots & v_{\lambda_n} \end{bmatrix}] = T$$

and since $T$ is nonsingular,

$$A = T\Delta T^{-1}$$

But then

$$A^k = T\Delta T^{-1}T\Delta T^{-1} \cdots T\Delta T^{-1} = T\Delta^k T^{-1} = T$$
The population model revisited

In our case:

\[
det(zI - A) = det \begin{bmatrix} z & -0.5 & 0 \\ -0.9 & z & 0 \\ 0 & -0.9 & z \end{bmatrix} = z^3 - \frac{9}{20}z = z(z - \frac{3}{2\sqrt{5}})(z + \frac{3}{2\sqrt{5}})
\]

Eigenvectors are found solving systems of equations

\[
Av_0 = 0 \cdot v_0 \implies v_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
Av_{\frac{3}{2\sqrt{5}}} = \frac{3}{2\sqrt{5}} \cdot v_{\frac{3}{2\sqrt{5}}} \implies v_{\frac{3}{2\sqrt{5}}} = \begin{bmatrix} \frac{5}{\sqrt{5}} \\ \frac{9}{3} \\ 1 \end{bmatrix}
\]

\[
Av_{-\frac{3}{2\sqrt{5}}} = -\frac{3}{2\sqrt{5}} \cdot v_{-\frac{3}{2\sqrt{5}}} \implies v_{-\frac{3}{2\sqrt{5}}} = \begin{bmatrix} \frac{5}{\sqrt{5}} \\ \frac{9}{3} \\ 1 \end{bmatrix}
\]
The population model revisited

Conclude

\[ A^k = \begin{bmatrix} \frac{5}{9} & \frac{5}{9} & 0 \\ \sqrt{\frac{5}{3}} & -\frac{\sqrt{5}}{3} & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2\sqrt{5}} & 0 & 0 \\ 0 & -\frac{3}{2\sqrt{5}} & 0 \\ 0 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} \frac{5}{9} & \frac{5}{9} & 0 \\ \sqrt{\frac{5}{3}} & -\frac{\sqrt{5}}{3} & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \]

Evolution of system is

\[ x(k) = \begin{bmatrix} \frac{5}{9} & \frac{5}{9} & 0 \\ \sqrt{\frac{5}{3}} & -\frac{\sqrt{5}}{3} & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2\sqrt{5}}^k & 0 & 0 \\ 0 & \left(-\frac{3}{2\sqrt{5}}\right)^k & 0 \\ 0 & 0 & 0^k \end{bmatrix} \begin{bmatrix} \frac{5}{9} & \frac{5}{9} & 0 \\ \sqrt{\frac{5}{3}} & -\frac{\sqrt{5}}{3} & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} x(0) \]

¿What if there is immigration?
The population model revisited

Model immigration as, say, \[
\begin{bmatrix}
0.2 \\
0.7 \\
0.1
\end{bmatrix}
\] \(u(k)\) where \(u(\cdot)\) is total number of immigrants. Then

\[
x(k) = \begin{bmatrix}
\frac{5}{2\sqrt{5}} & \frac{5}{2\sqrt{5}} & 0 \\
\frac{3}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & 0 \\
\frac{1}{3} & \frac{1}{3} & 1
\end{bmatrix}
\begin{bmatrix}
3^k \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\frac{5}{2\sqrt{5}} & \frac{5}{2\sqrt{5}} & 0 \\
\frac{3}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & 0 \\
\frac{1}{3} & \frac{1}{3} & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
x(0)
\end{bmatrix}
\]

\[
+ \sum_{j=0}^{k-1} \begin{bmatrix}
\frac{5}{2\sqrt{5}} & \frac{5}{2\sqrt{5}} & 0 \\
\frac{3}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & 0 \\
\frac{1}{3} & \frac{1}{3} & 1
\end{bmatrix}
\begin{bmatrix}
3^{k-1-j} \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\frac{5}{2\sqrt{5}} & \frac{5}{2\sqrt{5}} & 0 \\
\frac{3}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & 0 \\
\frac{1}{3} & \frac{1}{3} & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
x(0)
\end{bmatrix}
\]

\[
\cdot \begin{bmatrix}
\frac{9}{\sqrt{5}} & \frac{9}{\sqrt{5}} & 0 \\
\frac{5}{\sqrt{5}} & \frac{5}{\sqrt{5}} & 0 \\
\frac{1}{3} & \frac{1}{3} & 1
\end{bmatrix}
\begin{bmatrix}
0.2 \\
0.7 \\
0.1
\end{bmatrix}
u(j)
\]

A nightmare. Need help.
**\(\mathcal{Z}\)-transform analysis of state-space systems**

\(\mathcal{Z}\)-transform can be applied componentwise to sequences of vectors. By linearity, from state equation:

\[
\mathcal{Z}\left(\{x(k+1)\}\right) = A\mathcal{Z}\left(\{x(k)\}\right) + B\mathcal{Z}\left(\{u(k)\}\right)
\]
\( \mathcal{Z} \)-transform analysis of state-space systems

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Left-shift property:

\[
zX(z) - zx(0) = AX(z) + BU(z)
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\]

Left-shift property:

\[
zX(z) - zx(0) = AX(z) + BU(z)
\]

Conclude

\[
X(z) = z (zl_n - A)^{-1} x(0) + (zl_n - A)^{-1} BU(z)
\]

and

\[
Y(z) = zC (zl_n - A)^{-1} x(0) + \left[ C (zl_n - A)^{-1} B + D \right] U(z)
\]
By inspection easy to see that

\[(zI_n - A)^{-1} = \frac{1}{z} I_n + A \frac{1}{z^2} + \cdots + A^{k-1} \frac{1}{z^k} + \cdots\]

Substituting in previous formulas and inverse transforming we find again time-domain solution. Only, now we can work componentwise (many problems, but at least scalar ones!)

Rational matrix

\[H(z) = C (zI_n - A)^{-1} B + D\]

is transfer function of system.
Example

Compute $x(\cdot)$ corresponding to a step input when

$$x(k + 1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

and $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Answer: Can do convolution. However, note that

$$X(z) = z(zI - A)^{-1} x(0) + (zI - A)^{-1} bU(z)$$

Now

$$(zI - A)^{-1} = \begin{bmatrix} \frac{z+3}{z^2+3z+2} & \frac{1}{z} \\ \frac{-2}{z^2+3z+2} & \frac{1}{z} \end{bmatrix}$$

and consequently

$$X(z) = z \begin{bmatrix} \frac{1}{z^2+3z+2} \\ \frac{z}{z^2+3z+2} \end{bmatrix} + \begin{bmatrix} \frac{1}{z^2+3z+2} \\ \frac{z}{z^2+3z+2} \end{bmatrix} \frac{z}{z - 1}$$
We can now invert component-by-component:

\[
\frac{z}{z^2 + 3z + 2} = \frac{-1}{z + 1} + \frac{2}{z + 2} \therefore -\{(-1)^{k-1}\} + 2\{(-2)^{k-1}\}
\]

\[
\frac{z^2}{z^2 + 3z + 2} = 1 + \frac{1}{z + 1} + \frac{-4}{z + 2} \therefore \delta(\cdot) + \{(-1)^{k-1}\} - 4\{(-2)^{k-1}\}
\]

\[
\frac{1}{z^2 + 3z + 2} \frac{z}{z - 1} = \frac{1}{6} + \frac{1}{z + 1} + \frac{-2}{z + 2} \therefore \frac{1}{6}\{1^{k-1}\} + \frac{1}{2}\{(-1)^{k-1}\}
\]

\[
\frac{2}{3}\{(-2)^{k-1}\}
\]

\[
\frac{z}{z^2 + 3z + 2} \frac{z}{z - 1} = \frac{1}{6} + \frac{-1}{z + 1} + \frac{4}{3}{z + 2} \therefore \frac{1}{6}\{1^{k-1}\} - \frac{1}{2}\{(-1)^{k-1}\}
\]

\[
+\frac{4}{3}\{(-2)^{k-1}\}
\]

Summing the results up component-wise leads to the desired expression for \(x_1(k), x_2(k)\).
Discretization of state-space equations
Prolegomena: the matrix exponential

$$e^{At} = I_n + At + \cdots + \frac{1}{k!} A^k t^k + \cdots$$
Prolegomena: the matrix exponential

\[ e^{At} = I_n + At + \cdots + \frac{1}{k!} A^k t^k + \cdots \]

Notice that if \( A \) is a number, then this is the series representation of the exponential.

If \( A \) is diagonal, i.e. \( A = \text{diag}(\lambda_i) \), then \( e^{At} \) is also diagonal: \( e^{At} = \text{diag}(e^{\lambda_i t}) \).
Prolegomena: the matrix exponential

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If \( A \) is diagonal, i.e. \( A = \text{diag}(\lambda_i) \), then \( e^{At} \) is also diagonal: \( e^{At} = \text{diag}(e^{\lambda_i t}) \).

In order to compute it, assume that \( A \) has a basis of eigenvectors, then \( A = T \Lambda T^{-1} \) and consequently

\[
\begin{align*}
    e^{At} &= I_n + T \Lambda T^{-1} t + \cdots + \frac{1}{k!} T \Lambda^k T^{-1} t^k + \cdots \\
        &= T(I_n + \Lambda t + \cdots + \frac{1}{k!} \Lambda^k t^k + \cdots) T^{-1} \\
        &= T e^{\Lambda t} T^{-1} = T \text{diag}(e^{\lambda_i t}) T^{-1}
\end{align*}
\]
Prolegomena: the matrix exponential

\[ e^{At} = I_n + At + \cdots + \frac{1}{k!} A^k t^k + \cdots \]

Moreover,

\[ \frac{d}{dt} e^{At} = \frac{d}{dt} I_n + \frac{d}{dt} At + \cdots + \frac{d}{dt} \frac{1}{k!} A^k t^k + \cdots \]

\[ = A(I_n + At + \cdots + \frac{1}{k!} A^k t^k + \cdots) = Ae^{At} \]
Prolegomena: the matrix exponential

\[ e^{At} = I_n + At + \cdots + \frac{1}{k!} A^k t^k + \cdots \]

Moreover,

\[
\frac{d}{dt} e^{At} = \frac{d}{dt} I_n + \frac{d}{dt} At + \cdots + \frac{d}{dt} \frac{1}{k!} A^k t^k + \cdots \\
= A(I_n + At + \cdots + \frac{1}{k!} A^k t^k + \cdots) = Ae^{At}
\]

Note also \( e^{A \cdot 0} = I \).
Prolegomena: continuous-time state-space systems

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

Defining \( e^{At} := I_n + At + \cdots + \frac{1}{k!} A^k t^k + \cdots \), solution is

\[ x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \]
\[ y(t) = Ce^{At} x(0) + \int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t) \]

With starting time \( t_0 \):

\[ x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau \]

\( e^{At} \) is the all-important state transition matrix
Prolegomena: continuous-time state-space systems

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\dot{x} = Ax + Bu \\
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Prolegomena: the state transition matrix

\[ x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \]

If \( u(\cdot) = 0 \) in \([t_0, t]\), then

\[ x(t) = e^{A(t-t_0)}x(t_0) \]

i.e., the state transition matrix maps \( x(t_0) \) in \( x(t) \):

\[ x(t_0) \xrightarrow{e^{A(t-t_0)}} x(t) \]

Using STM one can “jump” from one state to the other.
Examples of continuous-time state-space systems

- RLC circuits (without loops including only capacitors or inductors) with state variable vector consisting of the voltages (charges) in the capacitors and the currents (fluxes) in the inductors;
Examples of continuous-time state-space systems

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- Mechanical systems with state variable being the “configuration variables” positions and velocities (or momenta);
Examples of continuous-time state-space systems

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- Mechanical systems with state variable being the “configuration variables” positions and velocities (or momenta);

- ...
Example

Find the response of the CT system

\[ \dot{x}(t) = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t) \]

to the step input \( \bar{1}(\cdot) \) when \( x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).
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Find the response of the CT system

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to the step input \( \bar{1}(\cdot) \) when \( x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

We must compute the exponential of the matrix \( A \). Observe that the eigenvector decomposition of \( A \) is

\[
\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}
\]
Example

Find the response of the CT system

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\]

Conclude that

\[
e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \left( \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^k \right) \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{0t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2e^t - 1 & 1 - e^t \\ 2e^t - 2 & 2 - e^t \end{bmatrix}
\]
Using the formula

\[ x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \]

we conclude that

\[
\begin{align*}
\begin{bmatrix} 1 \\ 2 \end{bmatrix} + & \int_0^t \begin{bmatrix} 2e^{t-\tau} - 1 & 1 - e^{t-\tau} \\ 2e^{t-\tau} - 2 & 2 - e^{t-\tau} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} d\tau \\
= & \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2t + 3(-1 + e^t) \\ -4t + 3(-1 + e^t) \end{bmatrix}
\end{align*}
\]
Discretization of CT state-space equations

At sampling times \((k + 1)T\) and \(kT\), \(k = 0, \ldots\) state is

\[
x((k + 1)T) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T - \tau)}Bu(\tau)d\tau
\]

\[
x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT - \tau)}Bu(\tau)d\tau
\]
Discretization of CT state-space equations

At sampling times $(k + 1)T$ and $kT$, $k = 0, \ldots$ state is

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x((k + 1)T) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau
\]

\[
x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau
\]

From this conclude

\[
x((k + 1)T) = e^{AT}x(kT) + e^{A(k+1)T} \int_k^{(k+1)T} e^{A(\tau-kT)}Bu(\tau)d\tau
\]

Now, assume that we use ZOH, i.e. that $u(\tau)$ is kept constant, equal to $u(kT)$, between $kT$ and $(k + 1)T$...
Discretization of CT state-space equations

ZOH implies \( u(t) = u(kT) \) for \( kT \leq t < (k + 1)T \):

\[
x((k + 1)T) = e^{AT}x(kT) + \int_0^T e^{A(T-\tau)}Bd\tau u(kT) =: G + H
\]
Discretization of CT state-space equations

ZOH implies $u(t) = u(kT)$ for $kT \leq t < (k + 1)T$:

$x((k + 1)T) = e^{AT}x(kT) + \int_{0}^{T} e^{A(T-\tau)} B d\tau u(kT)$

Conclude

$x((k + 1)T) = Gx(kT) + H u(kT)$
Discretization of CT state-space equations

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$$x((k + 1)T) = e^{AT}x(kT) + \int_0^T e^{A(T-\tau)}Bd\tau u(kT)$$

Conclude

$$x((k + 1)T) = Gx(kT) + Hu(kT)$$

Follows

$$y(kT) = Cx(kT) + Du(kT)$$

Note: $A \sim G$, $B \sim H$, $C$ and $D$ stay the same!
Example

Find ZOH equivalent for \( T = 0.1 \text{s} \) to

\[
\dot{x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]

Computation of exponential of \( A \equiv \) efficient computation of powers of \( A \) is key. Diagonalization of \( A \): \( A = V \Lambda V^{-1} \), with \( \Lambda = \text{diag}(\lambda_i) \), \( i \)-th column of \( V \) eigenvector associated with \( i \).

\[
A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \implies A^k = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}
\]
Example

Find ZOH equivalent for \( T = 0.1 \text{s} \) to

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\dot{x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]

Follows

\[
e^{At} = l_2 + At + \cdots + \left[ \begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array} \right] \left[ \begin{array}{cc} -1 & 0 \\ 0 & -2 \end{array} \right]^k \left[ \begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array} \right]^{-1} \frac{t^k}{k!} + \cdots
\]

\[
= \left[ \begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array} \right] \left[ l_2 + \cdots + \left[ \begin{array}{cc} -1 & 0 \\ 0 & -2 \end{array} \right]^k \frac{t^k}{k!} + \cdots \right] \left[ \begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array} \right]^{-1}
\]

\[
= \left[ \begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array} \right] \left[ \begin{array}{cc} e^{-t} & 0 \\ 0 & e^{-2t} \end{array} \right] \left[ \begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array} \right]^{-1} = \left[ \begin{array}{cc} 2e^{-2t} - e^{-t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-t} - e^{-2t} \end{array} \right]
\]
Example

Find ZOH equivalent for $T = 0.1\text{s}$ to

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

From this formula follows

$$G = e^{A0.1} = \begin{bmatrix} 2e^{-0.2} - e^{-0.1} & e^{-0.1} - e^{-0.2} \\ 2e^{-0.2} - 2e^{-0.1} & 2e^{-0.1} - e^{-0.2} \end{bmatrix} = \begin{bmatrix} 0.7326 & 0.0861 \\ -0.1722 & 0.9909 \end{bmatrix}$$

and

$$H = \int_0^{0.1} \begin{bmatrix} 2e^{-2(0.1-\tau)} - e^{-(0.1-\tau)} & e^{-(0.1-\tau)} - e^{-2(0.1-\tau)} \\ 2e^{-2(0.1-\tau)} - 2e^{-(0.1-\tau)} & 2e^{-(0.1-\tau)} - e^{-2(0.1-\tau)} \end{bmatrix} Bd\tau$$

$$= \begin{bmatrix} 0.004528 \\ 0.09969 \end{bmatrix}$$
Matlab for state-space discretization

Previous example, with $C = I_2$ and $D = 0$:

```matlab
sys = ss(A, B, C, D);
sysd = c2d(sys, 0.1, 'zoh')
```

\[
a = \\
\begin{array}{cccc}
x1 & x2 \\
x1 & 0.7326 & 0.08611 \\
x2 & -0.1722 & 0.9909 \\
\end{array}
\]

\[
b = \\
\begin{array}{c}
u1 \\
x1 & 0.004528 \\
x2 & 0.09969 \\
\end{array}
\]

\[
c = \\
\begin{array}{c}
x1 & x2 \\
y1 & 1 & 0 \\
y2 & 0 & 1 \\
\end{array}
\]

\[
d = \\
\begin{array}{c}
u1 \\
y1 & 0 \\
y2 & 0 \\
\end{array}
\]

Sampling time: 0.1
Discrete-time model.
An example-1

Oscillating mass without friction, position $q$:

$$\frac{d^2}{dt^2} q + q = 0$$
An example-1

Oscillating mass without friction, position $q$:

$$\frac{d^2}{dt^2} q + q = 0$$

Define $x = \begin{bmatrix} q \\ \frac{dq}{dt} \end{bmatrix}$, then state equations are

$$\frac{d}{dt} \begin{bmatrix} q \\ \frac{dq}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} q \\ \frac{dq}{dt} \end{bmatrix}$$
An example-1

Oscillating mass without friction, position \( q \):

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\frac{d^2}{dt^2} q + q = 0
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Define \( x = \begin{bmatrix} q \\ \frac{dq}{dt} \end{bmatrix} \), then state equations are

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\]

Observe \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \begin{bmatrix} -j & 0 \\ 0 & j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}^{-1} \)
An example-1

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Follows

\[
e^{At} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \begin{bmatrix} e^{-jt} & 0 \\ 0 & e^{jt} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}^{-1}
\]
Consequently

\[
e^{At} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \begin{bmatrix} e^{-jt} & 0 \\ 0 & e^{jt} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \begin{bmatrix} e^{-jt} & 0 \\ 0 & e^{jt} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}
\]

\[
= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}
\]
Consequently

\[ e^{At} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} e^{-jt} \begin{bmatrix} 0 & e^{jt} \\ e^{jt} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}^{-1} \]

\[ = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} e^{-jt} \begin{bmatrix} 1 & 1 \\ 0 & e^{jt} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} + \frac{j}{2} \\ \frac{1}{2} - \frac{j}{2} \end{bmatrix} \]

\[ = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \]

Evolution of the CT system:

\[ x(t) = e^{At} x(0) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0) \]
Evolution of the CT system:

\[ x(t) = e^{At} x(0) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0) \]

Evolution of the discretized system, sampling time \( T \):

\[ x[kT] = e^{AkT} x[0] = \begin{bmatrix} \cos kT & \sin kT \\ -\sin kT & \cos kT \end{bmatrix} x[0] \]
Evolution of the CT system:

\[ x(t) = e^{At} x(0) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0) \]

Evolution of the discretized system, sampling time \( T \):

\[ x[kT] = e^{A[kT]} x[0] = \begin{bmatrix} \cos kT & \sin kT \\ -\sin kT & \cos kT \end{bmatrix} x[0] \]

E.g. for \( x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) (unit displacement, zero initial velocity), discretized system is “photograph” of CT system at \( t_0 = 0, t_1 = T, t_2 = 2T, \ldots \)
System properties and discretization: controllability
System properties: controllability

Steering $x(\cdot)$ from arbitrary $x(0)$ to arbitrary $x_{\text{fin}}$
System properties: controllability

Steering $x(\cdot)$ from arbitrary $x(0)$ to arbitrary $x_{\text{fin}}$

$(A, B)$ is controllable if $\forall x(0)$ and $x_{\text{fin}}$ there exists a finite sequence $u(0), \ldots, u(N - 1)$ s.t.

$$x(N) = A^N x(0) + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j) = x_{\text{fin}}$$
System properties: controllability

Steering $x(\cdot)$ from arbitrary $x(0)$ to arbitrary $x_{\text{fin}}$

$(A, B)$ is controllable if $\forall x(0)$ and $x_{\text{fin}}$ there exists a finite sequence $u(0), \ldots, u(N - 1)$ s.t.

$$x(N) = A^N x(0) + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j) = x_{\text{fin}}$$

Equivalent (for $A$ nonsingular) with

$$\begin{align*}
\text{rank} \left( \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \right) &= n \\
\text{rank} \left( \begin{bmatrix} B \\ AB \\ \vdots \\ A^{n-1}B \\ \vdots \\ A^{n-1}B \end{bmatrix} \right) &= n
\end{align*}$$

¿Why?
Sketch of proof (single-input case)

Want to solve

\[ x(N) = x_{\text{fin}} = A^N x(0) + \sum_{j=0}^{N-1} A^{N-1-j} b u(j) \]

for some finite \( N \); equivalently,

\[ x_{\text{fin}} - A^N x(0) = \underbrace{[b \quad A b \quad \cdots \quad A^{N-1} b]}_{\text{arbitrary}} \begin{bmatrix} u(N-1) \\ \vdots \\ u(0) \end{bmatrix} \]
Sketch of proof (single-input case)

Want to solve

\[ x(N) = x_{\text{fin}} = A^N x(0) + \sum_{j=0}^{N-1} A^{N-1-j} bu(j) \]

for some finite \( N \); equivalently,

\[ x_{\text{fin}} - A^N x(0) = \begin{bmatrix} b & Ab & \cdots & A^{N-1} b \end{bmatrix} \begin{bmatrix} u(N-1) \\ \vdots \\ u(0) \end{bmatrix} \]

Possible if and only if

\[ \text{image} \left( \begin{bmatrix} b & Ab & \cdots & A^{N-1} b \end{bmatrix} \right) = \mathbb{R}^n \]

...can be shown (via Cayley-Hamilton theorem) that \( N \) can be taken \( \leq n \), dimension of the state.
Back to population example
A director of national park wants to achieve an arbitrary population distribution for the species with “autonomous” dynamics

\[
x(k + 1) = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.3 & 0 & 0 \\ 0 & 0.7 & 0.1 \end{bmatrix} x(k)
\]

where \( x(k) = \begin{bmatrix} y(k) \\ m(k) \\ v(k) \end{bmatrix} \). Can he/she use a management policy described by the input vector \( \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \), i.e. when the dynamics are

\[
x(k + 1) = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.3 & 0 & 0 \\ 0 & 0.7 & 0.1 \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} u(k)?
\]
Back to population example

Answer: No. For that $b$, the system is not controllable:

$$C(A, b) = \begin{bmatrix} -1 & \frac{3}{2} & -\frac{3}{20} \\ 3 & -\frac{3}{2} & \frac{9}{20} \\ 0 & \frac{21}{10} & \frac{20}{10} \end{bmatrix}$$

has rank 2, not 3.
Back to population example

Answer: No. For that \( b \), the system is not controllable:

\[
C(A, b) = \begin{bmatrix}
-1 & \frac{3}{2} & -\frac{3}{20} \\
3 & -\frac{3}{10} & \frac{9}{20} \\
0 & \frac{21}{10} & 0
\end{bmatrix}
\]

has rank 2, not 3.

Bambi is safe!

(Negative input for input vector \( B \) means: **shoot**!)
Back to population example

*Answer:* No. For that \( b \), the system is not controllable:

\[
C(A, b) = \begin{bmatrix}
-1 & 3 & -3 \\
3 & -\frac{3}{2} & -\frac{3}{20} \\
0 & -\frac{3}{10} & \frac{9}{20} \\
\end{bmatrix}
\]

has rank 2, not 3.

He/she can, however, reach any target population in less than or equal to 3 time steps if he/she uses a policy:

\[
\begin{bmatrix}
1 \\
3 \\
0 \\
\end{bmatrix}
\]

because then \((A, b)\) is controllable.
Warning

What follows is an **optional** part.

I will make clear where normal operations resume.
A digression on controllability: naïve optimal control
Consider the system

\[ x(k + 1) = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) ; \]

we want to reach \( x_{\text{fin}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \) from \( x(0) = 0 \).
Consider the system

\[ x(k + 1) = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) ; \]

we want to reach \( x_{\text{fin}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \) from \( x(0) = 0 \).

It is possible: \( C(A, B) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \) is nonsingular.
Example-1

Consider the system

\[ x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) ; \]

we want to reach \( x_{\text{fin}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \) from \( x(0) = 0 \).

Define cost of input sequence \( u(0), \ldots, u(N-1) \):

\[ \sum_{j=0}^{N-1} u(j)^2 ; \]

then since \( x_{\text{fin}} = C(A, B) \begin{bmatrix} -5 \\ 3 \end{bmatrix} \), cost for this \( u(\cdot) \) is \((-5)^2 + 3^2 = 34 \).
Example-2
Can we decrease the cost if we let ourselves one more time instant to reach $x_{fin}$?
Example-2

Can we decrease the cost if we let ourselves one more time instant to reach $x_{\text{fin}}$?

Inputs bringing $x(0) = 0$ to $x_{\text{fin}}$ are the solutions of

$$x_{\text{fin}} = \begin{bmatrix} b & Ab & A^2 b \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 9 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}. \quad (1)$$
Example-2

Can we decrease the cost if we let ourselves one more time instant to reach $x_{\text{fin}}$?

Inputs bringing $x(0) = 0$ to $x_{\text{fin}}$ are the solutions of

$$x_{\text{fin}} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = 0 . \quad (1)$$

All of these inputs are of the form: special solution of (1)+solution of the homogeneous equation

$$\begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} u'(2) \\ u'(1) \\ u'(0) \end{bmatrix} = 0 .$$
Example-3
So the general solution to our problem is

\[
\begin{bmatrix}
u_\alpha(2) \\
u_\alpha(1) \\
u_\alpha(0)
\end{bmatrix} = \begin{bmatrix}
-5 \\
3 \\
0
\end{bmatrix} + \begin{bmatrix}
3 \\
-4 \\
1
\end{bmatrix} \alpha
\]
Example-3
So the general solution to our problem is

\[
\begin{bmatrix}
  u_\alpha(2) \\
  u_\alpha(1) \\
  u_\alpha(0)
\end{bmatrix} =
\begin{bmatrix}
  -5 \\
  3 \\
  0
\end{bmatrix} +
\begin{bmatrix}
  3 \\
  -4 \\
  1
\end{bmatrix} \alpha
\]

The (\(\alpha\)-dependent) cost of this input is

\[34 - 54\alpha + 26\alpha^2\]

which takes its minimum for \(\alpha = 1.03846\). The cost is 5.96154.
Example-3
So the general solution to our problem is

\[
\begin{bmatrix}
  u_\alpha(2) \\
  u_\alpha(1) \\
  u_\alpha(0)
\end{bmatrix}
= \begin{bmatrix}
  -5 \\
  3 \\
  0
\end{bmatrix}
+ \begin{bmatrix}
  3 \\
  -4 \\
  1
\end{bmatrix} \alpha
\]

The (\(\alpha\)-dependent) cost of this input is

\[
34 - 54\alpha + 26\alpha^2
\]

which takes its minimum for \(\alpha = 1.03846\). The cost is 5.96154.

WOW! By allowing ourselves one time instant more to reach \(x_{\text{fin}}\) we got there with a lot less cost!
Example-4
With 4 time instants, general input leading to $x_{fin}$ is

$$
\begin{bmatrix}
u_{\alpha,\beta}(3) \\
u_{\alpha,\beta}(2) \\
u_{\alpha,\beta}(1) \\
u_{\alpha,\beta}(0)
\end{bmatrix} = \begin{bmatrix}
-5 \\
3 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
3 \\
-4 \\
1 \\
0
\end{bmatrix} \alpha + \begin{bmatrix}
12 \\
-13 \\
0 \\
1
\end{bmatrix} \beta
$$
Example-4

With 4 time instants, general input leading to $x_{\text{fin}}$ is

$$\begin{bmatrix} u_{\alpha,\beta}(3) \\ u_{\alpha,\beta}(2) \\ u_{\alpha,\beta}(1) \\ u_{\alpha,\beta}(0) \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \\ 1 \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} 12 \\ -13 \\ 0 \\ 1 \end{bmatrix} \beta$$

Cost is

$$\alpha^2 + \beta^2 + (-5 + 3\alpha + 12\beta)^2 + (-3 + 4\alpha + 13\beta)^2$$

with minimum at $\alpha = -0.557143, \beta = 0.471429$ equal to 2.37143.
**Example-4**

With 4 time instants, general input leading to $x_{\text{fin}}$ is

$$
\begin{bmatrix}
  u_{\alpha,\beta}(3) \\
  u_{\alpha,\beta}(2) \\
  u_{\alpha,\beta}(1) \\
  u_{\alpha,\beta}(0)
\end{bmatrix} =
\begin{bmatrix}
  -5 \\
  3 \\
  0 \\
  0
\end{bmatrix} +
\begin{bmatrix}
  3 \\
  -4 \\
  1 \\
  0
\end{bmatrix} \alpha +
\begin{bmatrix}
  12 \\
  -13 \\
  0 \\
  1
\end{bmatrix} \beta
$$

Cost is

$$
\alpha^2 + \beta^2 + (-5 + 3\alpha + 12\beta)^2 + (-3 + 4\alpha + 13\beta)^2
$$

with minimum at $\alpha = -0.557143, \beta = 0.471429$ equal to 2.37143.

Not surprisingly, we decreased the cost...
If we leave ourselves one more time instant, we cannot do worse than this...
Example-5

If we leave ourselves one more time instant, we cannot do worse than this...

...of course, we’ll have a 3-parameter optimization problem to solve.
Example-5

If we leave ourselves one more time instant, we cannot do worse than this...

...of course, we’ll have a 3-parameter optimization problem to solve.

- Does there exist an absolute minimum, e.g. if we let ourselves infinite time?
Example-5

If we leave ourselves one more time instant, we cannot do worse than this...

...of course, we’ll have a 3-parameter optimization problem to solve.

- Does there exist an absolute minimum, e.g. if we let ourselves infinite time?

- Do we have to go through this excruciating procedure of symbolic computations to get the answer?
Example-5

If we leave ourselves one more time instant, we cannot do worse than this...

...of course, we’ll have a 3-parameter optimization problem to solve.

- Does there exist an absolute minimum, e.g. if we let ourselves infinite time?
- Do we have to go through this excruciating procedure of symbolic computations to get the answer?
- It seems that every time we pad the previous candidate \( u(\cdot) \) with zero, then “add one term” to it. Recursion?
If we leave ourselves one more time instant, we cannot do worse than this...

...of course, we’ll have a 3-parameter optimization problem to solve.

- Does there exist an absolute minimum, e.g. if we let ourselves infinite time?

- Do we have to go through this excruciating procedure of symbolic computations to get the answer?

- It seems that every time we pad the previous candidate $u(\cdot)$ with zero, then “add one term” to it. Recursion?

- What if we would have a cost depending also on $x(\cdot)$, say to prevent “jerks” on $x(\cdot)$ during the transit?
Moral of the story

Let go of the illusion of “control asap”. Take your time. Focus your mind on energy.
Moral of the story

Let go of the illusion of “control asap”.
Take your time. Focus your mind on energy.

Optimal control
Warning

From now on, normal operations resume.

What follows is not optional anymore!
System properties and discretization: observability
Determining $x(0)$ from $u(\cdot), y(\cdot)$
Observability

Determining $x(0)$ from $u(\cdot), y(\cdot)$

$(A, c)$ is observable if $\forall u(\cdot), y(\cdot)$, there exists a unique $x(0)$ such that

$$y(k) = cA^k x(0) + \sum_{j=0}^{k-1} cA^{k-1-j} bu(j) + du(k)$$
Observability

Determining $x(0)$ from $u(\cdot), y(\cdot)$

$(A, c)$ is observable if $\forall u(\cdot), y(\cdot)$, there exists a unique $x(0)$ such that

$$y(k) = cA^k x(0) + \sum_{j=0}^{k-1} cA^{k-1-j} bu(j) + du(k)$$

Equivalent with

$$\text{rank} \begin{pmatrix} c \\ \vdots \\ cA^{n-1} \end{pmatrix} = n$$
Sketch of proof

\[ y(0) = cx(0) + du(0) \]
\[ y(1) = cAx(0) + cbu(0) + du(1) \]
\[ \vdots \]
\[ y(k) = cA^k x(0) + \sum_{j=0}^{k-1} cA^{k-1-j} bu(j) + du(k) \]

equivalent with

\[
\begin{bmatrix}
    y(0) - du(0) \\
    y(1) - cbu(0) - du(1) \\
    \vdots \\
    y(k) - \sum_{j=0}^{k-1} cA^{k-1-j} bu(j) - du(k)
\end{bmatrix}
\begin{bmatrix}
    c \\
    cA \\
    \vdots \\
    cA^{k-1}
\end{bmatrix}

= \begin{bmatrix} x(0) \end{bmatrix}
\]

Easy to see that it is solvable for all \( x(0) \) if and only if \( \text{rank}(O(A, c)) = n \).
Exercise

Consider the system

\[
x(k + 1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)
\]

\[
y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \end{bmatrix} u(k)
\]

Assume that in response to \( u(\cdot) = \{1, 1, 0, 0, \ldots\} \), \( y(\cdot) = \{2, 3, -4, 11, -25, \ldots\} \). What is \( x(0) \)?
Exercise

Consider the system

\[
x(k + 1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)
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y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \end{bmatrix} u(k)
\]

Assume that in response to \( u(\cdot) = \{1, 1, 0, 0, \cdots \} \), \( y(\cdot) = \{2, 3, -4, 11, -25, \cdots \} \). What is \( x(0) \)?

Recall

\[
y(k) = CA^k x(0) + \sum_{j=0}^{k-1} CA^{k-1-j} Bu(j) + Du(k):
\]

\[
y(0) = Cx(0) + Du(0)
\]

\[
y(1) = CAx(0) + CBu(0) + Du(1)
\]

\[\vdots = \vdots\]
Exercise

Consider the system

\[
x(k + 1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)
\]

\[
y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \end{bmatrix} u(k)
\]

Assume that in response to \( u(\cdot) = \{1, 1, 0, 0, \cdots\} \), \( y(\cdot) = \{2, 3, -4, 11, -25, \cdots\} \). What is \( x(0) \)?

It follows

\[
2 = x_1(0) + 1 \cdot 1
\]
\[
3 = x_2(0) + 1 \cdot 1 + 1 \cdot 1
\]
\[
\vdots
\]

so \( x_1(0) = 1, x_2(0) = 1 \).
Consider the system

\[
x(k + 1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)
\]

\[
y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \end{bmatrix} u(k)
\]

Assume that in response to \( u(\cdot) = \{1, 1, 0, 0, \ldots\} \), \( y(\cdot) = \{2, 3, -4, 11, -25, \ldots\} \). What is \( x(0) \)?

It follows

\[
2 = x_1(0) + 1 \cdot 1
\]
\[
3 = x_2(0) + 1 \cdot 1 + 1 \cdot 1
\]
\[
\vdots = \vdots
\]

so \( x_1(0) = 1 \), \( x_2(0) = 1 \).

Adding yet another equation does not help in determining \( x(0) \) uniquely (Cayley-Hamilton).
Example-1
Consider the system without inputs described by

\[
x(k + 1) = \begin{bmatrix} -1 & 0 & \alpha \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x(k)
\]

\[
y(k) = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} x(k).
\]

For which values of \( \alpha \) is the system observable?
Example-1
Consider the system without inputs described by

\[
x(k + 1) = \begin{bmatrix}
-1 & 0 & \alpha \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} x(k)
\]

\[
y(k) = \begin{bmatrix}
0 & 1 & 2
\end{bmatrix} x(k)
\]

For which values of \( \alpha \) is the system observable? We compute the observability matrix

\[
O(A, c) = \begin{bmatrix}
c \\
cA \\
cA^2
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 2 \\
0 & -1 & -2 \\
0 & 1 & 2
\end{bmatrix}
\]
Example-1

Consider the system without inputs described by

\[
x(k + 1) = \begin{bmatrix} -1 & 0 & \alpha \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x(k)
\]

\[
y(k) = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} x(k) .
\]

For which values of $\alpha$ is the system observable?

We compute the observability matrix

\[
\mathcal{O}(A, c) = \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} .
\]

Now $\det \mathcal{O}(A, c) = 0$. The matrix is singular for all $\alpha$! The system is never observable!
What does this mean, in practice?
Example-2

What does this mean, in practice?

Observe that for all $\alpha$

$$\ker \mathcal{O}(A, c) = \{ v \in \mathbb{R}^3 \mid \mathcal{O}(A, c)v = 0 \}$$

$$= \left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \beta + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \gamma \mid \beta, \gamma \in \mathbb{R} \right\}$$
What does this mean, in practice?

Observe that for all $\alpha$

$$\ker \mathcal{O}(A, c) = \{ v \in \mathbb{R}^3 \mid \mathcal{O}(A, c)v = 0 \}$$

$$= \left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \beta + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \gamma \mid \beta, \gamma \in \mathbb{R} \right\}$$

This means that $x(0)$ and $x(0) + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \beta + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \gamma$

generate the same response. It is impossible to distinguish them looking at $y(\cdot)$. 
Remarks on controllability/observability

- **Controllability**: ability to steer the state of the system wherever we want;

- Observability less intuitive, but often state is impossible or too costly to measure directly, so ability to infer it from input and output sequences is crucial;

- Structural properties: they depend on $A$, $b$, $c$ and the relation between them in a geometric sense (ranks of controllability/observability matrices);

- Analogous properties can be formulated for continuous-time state-space systems. Surprisingly, the geometric conditions are exactly the same!
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- **Structural** properties: they depend on $A$, $b$, $c$ and the relation between them in a geometric sense (ranks of controllability/observability matrices);

- Analogous properties can be formulated for continuous-time state-space systems. Surprisingly, the geometric conditions are exactly the same!
System properties under discretization

**Stability**: Since $G = e^{AT}$, the eigenvalues $\lambda_i$ of $A$ are mapped to eigenvalues $e^{\lambda_i T}$ of $G$. So, $\text{CT stability} \implies \text{DT stability}$. 

¿...what about controllability and observability?
When dealing with CT $\sim$ DT, necessary that continuous-time system is controllable/observable.

However, discretization with “bad” sampling can create loss of these properties.
Example: harmonic oscillator

Harmonic oscillator $m\ddot{x} + kw = 0$ in state space form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

observable.
Example: harmonic oscillator

Since

\[
A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \sqrt{\frac{k}{m}i} & -\sqrt{\frac{k}{m}i} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{k}{m}i} & 0 \\ 0 & -\sqrt{\frac{k}{m}i} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \sqrt{\frac{k}{m}i} & -\sqrt{\frac{k}{m}i} \end{bmatrix}
\]

it follows that

\[
G_T = \begin{bmatrix} \cos \left( \sqrt{\frac{k}{m}T} \right) & \sin \left( \sqrt{\frac{k}{m}T} \right) \sqrt{mk} \\ -\frac{k}{m} \sin \left( \sqrt{\frac{k}{m}T} \right) & \cos \left( \sqrt{\frac{k}{m}T} \right) \end{bmatrix}
\]
Example: harmonic oscillator

Choose $T = \sqrt{\frac{m}{k}} \pi$, then

$$G_T = e^{AT} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

which yields a non-observable discrete-time system

$$x(k + 1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$
Example: harmonic oscillator

Choose $T = \sqrt{\frac{m}{k}} \pi$, then

$$G_T = e^{AT} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

which yields a non-observable discrete-time system

$$x(k + 1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

We “photograph” the system at time intervals an oscillation period apart. Impossible to distinguish initial states.
Maintaining controllability and observability

Controllable and observable

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

with \( A \) having eigenvalues \( \lambda_i \).

**CT** Controllability and observability
\[ \implies \] **DT** controllability and observability if

\[ \frac{1}{T} > \frac{|\text{im}(\lambda_i)|}{\pi} \text{ for } i = 1, \ldots, n \]
Maintaining controllability and observability

Controllable and observable

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

with \( A \) having eigenvalues \( \lambda_i \).

CT Controllability and observability

\( \implies \) DT controllability and observability if

\[
\frac{1}{T} > \frac{|\text{im}(\lambda_i)|}{\pi} \quad \text{for} \quad i = 1, \ldots, n
\]

If the sampling frequency is greater than twice the largest frequency of the system (Nyquist frequency), then controllability and observability are maintained.
Maintaining controllability and observability

Controllable and observable

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

with \( A \) having eigenvalues \( \lambda_i \).

CT Controllability and observability
\[ \implies \] DT controllability and observability if

\[ \frac{1}{T} > \frac{|\text{im}(\lambda_i)|}{\pi} \]

for \( i = 1, \ldots, n \)

If the sampling frequency is greater than twice the largest frequency of the system (Nyquist frequency), then controllability and observability are maintained.

If \( \lambda_i \in \mathbb{R} \) for \( i = 1, \ldots, n \), then condition always true.
Maintaining controllability and observability

Controllable and observable

\[ \dot{x} = Ax + Bu \]

\[ y = Cx + Du \]

with \( A \) having eigenvalues \( \lambda_i \).

CT Controllability and observability
\[ \iff \] DT controllability and observability if

\[ \frac{1}{T} > \frac{|\text{im}(\lambda_i)|}{\pi} \text{ for } i = 1, \ldots, n \]

Loss of controllability and observability \( \equiv \) pole-zero cancellation in the transfer function...
Pole-zero cancellations

Consider

\[ x(k + 1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \]

\[ y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k) \]

Easy to see system is controllable, but not observable. Observe also that

\[ c(zI - A)^{-1} b = \frac{1}{z + 2} \]

¡A first order transfer function!
Pole-zero cancellations

Consider also

\[ x(k + 1) = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \]

\[ y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \]

which is observable but not controllable. ¡Same transfer function as before!
Intersample behaviour

Maybe good *sampled* behaviour, but anything may happen between samples.

¿How to compute what happens between samples?
Intersample behaviour

Due to ZOH, input to plant constant in \([kT, (k + 1)T]\).

Let \(N \geq 1\); then \(u(t) = \text{const}\) also between \(k\frac{T}{N}\) and \((k + 1)\frac{T}{N}\).
Intersample behaviour

If model is state-space, discretizing with step \( \frac{T}{N} \):

\[
G_1 = e^{A \frac{T}{N}}
\]

\[
H_1 = \int_0^{\frac{T}{N}} e^{A \left( \frac{T}{N} - \tau \right)} B d\tau
\]

Now run \( x(k+1) = G_1 x(k) + H_1 \overline{u}(k) \) where

\[
\overline{u}(k) = u(\text{int}(\frac{k}{N}))
\]

with \( \text{int}(\alpha) \) is the integer part of \( \alpha \).

\( x(k) \) is the value of the continuous-time state vector at time \( k \frac{T}{N} \), as we wanted.
Intersample behaviour

In this way, if there is loss of observability, we can obviate the problem by effectively re-sampling.
Intersample behaviour

In this way, if there is loss of observability, we can obviate the problem by effectively re-sampling.

Intersample ‘ripple’ (oscillatory behavior between samples) is caused by the controller canceling unstable or poorly damped zeroes of the plant. That is a problem of design, not solvable by re-sampling. However, re-sampling as above allows to check for intersample ripple.
Control via state feedback
State-feedback vs. output feedback

ELEC2220: output (sometimes, derivatives) fed back
State-feedback vs. output feedback

ELEC2220: output (sometimes, derivatives) fed back

Now, we feed linear function of state back ($K \in \mathbb{R}^{m \times n}$):

\[
\begin{align*}
    x(k+1) &= Ax(k) + Bu(k) \\
    y(k) &= Cx(k) + Du(k)
\end{align*}
\]

\[\leadsto\]

\[
\begin{align*}
    x(k+1) &= (A - BK)x(k) + Bv(k) \\
    y(k) &= Cx(k) + Dv(k)
\end{align*}
\]

and $u(k) = -Kx(k) + v(k)$
State-feedback vs. output feedback

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...transfer function is modified:

$$
\begin{align*}
    H(z) &= C(zI - A)^{-1}B + D \\
    &= C \frac{\text{Adj}(zI-A)}{\text{det}(zI - A)} B + D
\end{align*}
$$

$$
\begin{align*}
    H'(z) := C(zI - A + BK)^{-1}B + D \\
    &= C \frac{\text{Adj}(zI-A+BK)}{\text{det}(zI - A + BK)} B + D
\end{align*}
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\begin{align*}
    H'(z) := C(zI - A + BK)^{-1}B + D \\
    &= C \frac{\text{Adj}(zI-A+BK)}{\det(zI - A + BK)} B + D
\end{align*}
\]

...poles (roots of $\det(pl - A) = 0$) modified to roots of $\det(pl - A + BK) = 0$
R.E. Kalman proved that

If system is controllable, then for every choice of poles $p_i = 1, \ldots, n$ there exists $K \in \mathbb{R}^{m \times n}$ such that

$$x(k + 1) = (A - BK)x(k) + Bv(k)$$
$$y(k) = Cx(k) + Dv(k)$$

has poles at $p_i$, $i = 1, \ldots, n$
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\end{align*}
$$

has poles at $p_i, i = 1, \ldots, n$

under controllability, linear state feedback $\implies$ arbitrary pole placement!
Example

\[
x(k + 1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)
\]
\[
y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)
\]

...controllable, but bad poles (-1,-2).
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Define \( K := \begin{bmatrix} k_0 & k_1 \end{bmatrix} \). Then

\[ A - BK = \begin{bmatrix} 0 & 1 \\ -2 - k_0 & -3 - k_1 \end{bmatrix}, \]

with determinant \( z^2 + (-3 - k_1)z + (-2 - k_0) \).
Example

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x(k + 1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)
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Want poles at \( \frac{1}{2}, \frac{1}{3} \).

Imposing

\[
z^2 + (-3 - k_1)z + (-2 - k_0)\mid_{z=\frac{1}{2}} = \frac{1}{4} (-4k_0 - 2k_1 - 13) = 0
\]

\[
z^2 + (-3 - k_1)z + (-2 - k_0)\mid_{z=\frac{1}{3}} = \frac{1}{9} (-9k_0 - 3k_1 - 26) = 0,
\]

and solving for \( k_0, k_1 \), yields \( k_0 = k_1 = -\frac{13}{6} \).
Example

\[ x(k + 1) = \begin{bmatrix} 2 & 6 \\ -2 & -5 \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(k) \]

\[ y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} x(k) \]

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Example

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\end{align*}
\]

...controllable, but bad poles (-1,-2).

Want poles at \(\frac{1}{2}, \frac{1}{3}\).

Define \(K := \begin{bmatrix} k_0 & k_1 \end{bmatrix}\). Then

\[
A - BK = \begin{bmatrix}
k_0 + 2 & k_1 + 6 \\
-k_0 - 2 & -k_1 - 5
\end{bmatrix},
\]

with determinant \((k_0 + 2) + (k_1 + 3 - k_0)z + z^2\).
Example

\[
x(k + 1) = \begin{bmatrix} 2 & 6 \\ -2 & -5 \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(k)
\]
\[
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\]

...controllable, but bad poles (-1,-2).

Want poles at \( \frac{1}{2}, \frac{1}{3} \).

Imposing

\[
(k_0 + 2) + (k_1 + 3 - k_0)z + z^2 \bigg|_{z = \frac{1}{2}} = \frac{1}{4} (2k_0 + 2k_1 + 15) = 0
\]
\[
(k_0 + 2) + (k_1 + 3 - k_0)z + z^2 \bigg|_{z = \frac{1}{3}} = \frac{1}{9} (6k_0 + 3k_1 + 28) = 0 ,
\]

and solving for \( k_0, k_1 \), yields \( k_0 = -\frac{11}{6}, k_1 = -\frac{17}{3} \).
Many state-space realisations for a given TF:

\[
\frac{1}{z + \frac{1}{2}}
\]

realised by

\[
\begin{align*}
x(k + 1) &= -\frac{1}{2} x(k) + 1 \cdot u(k) \\
y(k) &= 1 \cdot x(k)
\end{align*}
\]

(unidimensional, internally stable)
Minimality

Many state-space realisations for a given TF:

\[
\frac{1}{z + \frac{1}{2}}
\]

and by

\[
x'(k + 1) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} x'(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)
\]

\[
y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x'(k)
\]

(bi-dimensional, internally unstable)
Minimality

*Many* state-space *realisations* for a given TF:

\[
\frac{1}{z + \frac{1}{2}}
\]

and by many, many more...
Minimality

**Minimal realisations**: those using the *minimal* number of state variables.

*Theorem* (Kalman) Equivalent:

1. \((A, B, C, D)\) is minimal realisation of 
   \[ H(z) = C(zI - A)^{-1}B + D \]

2. \(H(z) = C(zI - A)^{-1}B + D\) and \((A, B)\) is controllable and \((C, A)\) is observable

3. (SISO case) \(H(z) = C(zI - A)^{-1}B + D\) and
   \# state variables=degree of denominator of \(H(z)\), in **simplified** form.
Minimality and stability
Statement 3 of Kalman’s theorem: no controllability or observability $\iff$ no minimality.
Minimality and stability

Statement 3 of Kalman’s theorem: no controllability or observability $\implies$ no minimality.

*Internal instability* of non-minimal realisation may result in *BIBO-stability*
Minimality and stability

Statement 3 of Kalman’s theorem: no controllability or observability ⇒ no minimality.

*Internal instability* of non-minimal realisation may result in *BIBO-stability*

E.g. for \( \frac{1}{z + \frac{1}{2}} \)

\[
x'(k + 1) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 10 \end{bmatrix} x'(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)
\]

\[
y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x'(k)
\]
Observers
The reason for observers

State feedback regulation \((u(k) = -Kx(k))\) is based on the assumption that \(x(\cdot)\) is measurable.

However, often only \(u(\cdot)\) and \(y(\cdot)\) are known!
The reason for observers

State feedback regulation \( u(k) = -Kx(k) \) is based on the assumption that \( x(\cdot) \) is measurable.

However, often only \( u(\cdot) \) and \( y(\cdot) \) are known!

... for example it may be too costly/impossible to access \( x(\cdot) \)...

State feedback regulation \((u(k) = -Kx(k))\) is based on the assumption that \(x(\cdot)\) is measurable.

However, often only \(u(\cdot)\) and \(y(\cdot)\) are known!

¿Is it still possible to use state-feedback, even when \(x(\cdot)\) is not directly available?
The reason for observers

State feedback regulation \( u(k) = -Kx(k) \) is based on the assumption that \( x(\cdot) \) is measurable.

However, often only \( u(\cdot) \) and \( y(\cdot) \) are known!

¿Is it still possible to use state-feedback, even when \( x(\cdot) \) is not directly available?

Leads to state estimation.
Observers

Basic idea: construct a new system $(A', B', C', D')$ having as inputs

- the input $u(\cdot)$; and
- the output $y(\cdot)$

of original system, that outputs a vector $x'$ such that

$$\lim_{{k \to \infty}} x'(k) - x(k) =: \lim_{{k \to \infty}} e(k) = 0$$

$x'(\cdot)$ is called the state estimate, and $e(\cdot) = x'(\cdot) - x(\cdot)$ is called the estimate error.
Observers

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\]

\(x'(\cdot)\) is called the state estimate, and \(e(\cdot) = x'(\cdot) - x(\cdot)\) is called the estimate error.

Such a system \((A', B', C', D')\) is called a state observer.
Graphically...

\[ x(k+1) = Ax(k) + Bu(k) \]

\[ x'(k+1) = Ax'(k) + Bu(k) \]

\[ y(k) = Cx(k) \]

\[ e(k) = y'(k) - \hat{y}(k) \]
If \((A, C)\) is observable, then there exists \(L \in \mathbb{R}^{n \times p}\) such that the observer matrices can be chosen as

\[
A' = A - LC \\
B' = B \\
C' = C \\
D' = D
\]

These matrices define the equations of the observer.
If \((A, C)\) is observable, then there exists \(L \in \mathbb{R}^{n \times p}\) such that the observer matrices can be chosen as

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\begin{align*}
A' &= A - LC \\
B' &= B \\
C' &= C \\
D' &= D
\end{align*}
\]

These matrices define the equations of the observer.
Sketch of proof

Consider that if $A' = A - LC$ and $C' = C$, then the dynamics of the observer and of the plant respectively are

$$x'(k + 1) = Ax'(k) + Bu(k) + L(y(k) - Cx'(k))$$

and

$$x(k + 1) = Ax(k) + Bu(k)$$
Sketch of proof

Consider that if $A' = A - LC$ and $C' = C$, then the dynamics of the observer and of the plant respectively are

$$x'(k + 1) = Ax'(k) + Bu(k) + L(y(k) - Cx'(k))$$

and

$$x(k + 1) = Ax(k) + Bu(k)$$

Subtract the first equation from the second, and obtain

$$e(k + 1) = x(k) - x'(k)$$

$$= Ax(k) - Ax'(k) - L(Cx(k) - Cx'(k))$$

$$= (A - LC)(x(k) - x'(k)) = (A - LC)e(k)$$
Now consider that

\((A, C)\) observable if and only if \((A^\top, C^\top)\) controllable
Sketch of proof

Now consider that

\((A, C)\) observable if and only if \((A^\top, C^\top)\) controllable

We can thus assign the eigenvalues of \(A^\top - C^\top L^\top\) arbitrarily by suitably choosing the coefficients of \(L^\top\).
Sketch of proof

Now consider that

\((A, C)\) observable if and only if \((A^\top, C^\top)\) controllable

We can thus assign the eigenvalues of \(A^\top - C^\top L^\top\) arbitrarily by suitably choosing the coefficients of \(L^\top\).

But these are also the eigenvalues of \(A - LC\)!

\(\exists L\) so that the characteristic polynomial of \(A - LC\) to be stable, i.e. so that \(e(k) \to 0\) as \(k \to \infty\).
Original system:

\[
x(k + 1) = \begin{bmatrix} -1.908 & -0.9902 & -0.075 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k)
\]

\[
y(k) = \begin{bmatrix} 0 & 1 & -\frac{1}{4} \end{bmatrix} x(k)
\]
Example-1

Original system:

\[
x(k+1) = \begin{bmatrix} -1.908 & -0.9902 & -0.075 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k)
\]

\[
y(k) = \begin{bmatrix} 0 & 1 & -\frac{1}{4} \end{bmatrix} x(k)
\]

State estimate with observer poles at \(-\frac{2}{3}, -\frac{3}{4}, -\frac{4}{5}\), i.e.

\[
L = \begin{bmatrix} 0.0618 \\ 0.0330 \\ -1.1043 \end{bmatrix}
\]
Example-2

den=poly([-11/12 -9/10 -1/11]);
systf=tf(num,den,-1);
[A,b,c,d]=tf2ss(num,den);
sysss=ss(A,b,c,d,-1);
u=randn(100,1);
x0=rand(3,1);
[y,t,x]=lsim(sysss,u,1:100,x0);
des=[-2/3 -3/4 -4/5];
L=place(A',c',des);
x0l=4*rand(3,1);
observ=ss(A-L'*c,[b L'],c,d,-1);
[y1,t,x1]=lsim(observ,[u y],1:100,x0l);
plot(t,x(:,1),'r');
hold on
plot(t,x1(:,1),'b');
hold off
title('First component of original (red) and of estimated (blue) state');
figure
plot(t,x(:,2),'r');
hold on
plot(t,x1(:,2));
hold off
title('Second component of original (red) and of estimated (blue) state');
figure
plot(t,x(:,3),'r');
hold on
plot(t,x1(:,3));
hold off
title('Third component of original (red) and of estimated (blue) state');
Example-3

Third component of original (red) and of estimated (blue) state

Second component of original (red) and of estimated (blue) state

First component of original (red) and of estimated (blue) state
Deadbeat observers

\((A, C)\) observable \(\iff\) can assign all eigenvalues of \(A - LC\) to zero.

In such case, we talk about a deadbeat observer.
Deadbeat observers

$(A, C)$ observable $\implies$ can assign all eigenvalues of $A - LC$ to zero.

In such case, we talk about a deadbeat observer.

Linear Algebra: if $X \in \mathbb{R}^{n \times n}$ has only zero eigenvalues then there exists $k \leq n$ such that $X^k = 0$ (nilpotency).
Deadbeat observers

(A, C) observable \iff can assign all eigenvalues of \( A - LC \) to zero.

In such case, we talk about a **deadbeat observer**.

Linear Algebra: if \( X \in \mathbb{R}^{n \times n} \) has only zero eigenvalues then there exists \( k \leq n \) such that \( X^k = 0 \) (nilpotency). 

\[ e(j) = x(j) - x'(j) = 0 \quad \text{for } j \geq n! \]
Deadbeat observers

\((A, C)\) observable \(\implies\) can assign all eigenvalues of \(A - LC\) to zero.

In such case, we talk about a deadbeat observer.

Linear Algebra: if \(X \in \mathbb{R}^{n \times n}\) has only zero eigenvalues then there exists \(k \leq n\) such that \(X^k = 0\) (nilpotency).

\(\implies\) for deadbeat observers

\[ e(j) = x(j) - x'(j) = 0 \quad \text{for } j \geq n! \]

¡Finite-time exact reconstruction of state!
Putting it all together: observer-based control schemes
Observer-based feedback control

\[
x(k+1) = Ax(k) + Bu(k)
\]

\[
x'(k+1) = Ax'(k) + Bu(k)
\]

\[
y' = -K e
\]
Observer-based feedback control

In mathematics:

\[ x(k + 1) = Ax(k) + B(v(k) - Kx'(k)) \]
\[ x'(k + 1) = Ax'(k) + Bv(k) + L(y'(k) - Cx(k)) \]
\[ y(k) = Cx(k) \]
\[ y'(k) = Cx'(k) \]

...a system with state \( \begin{bmatrix} x \\ x' \end{bmatrix} \)
Observer-based feedback control

In mathematics:

\[
\begin{align*}
x(k + 1) &= Ax(k) + B(v(k) - Kx'(k)) \\
x'(k + 1) &= Ax'(k) + Bv(k) + L(y'(k) - Cx(k)) \\
y(k) &= Cx(k) \\
y'(k) &= Cx'(k)
\end{align*}
\]

...a system with state \[
\begin{bmatrix}
x \\ x'
\end{bmatrix}
\]

Easier to use \[
\begin{bmatrix}
x \\ x - x'
\end{bmatrix}
\] as state for analysis...
Observer-based feedback control

Equations become

\[ x(k + 1) = Ax(k) + B(v(k) - Kx(k)) + B(Kx(k) - Kx'(k)) \]
\[ = (A - BK)x(k) + Bv(k) + BK e(k) \]
\[ e(k + 1) = (A - LC)e(k) \]

i.e.

\[
\begin{bmatrix}
  x(k + 1) \\
  e(k + 1)
\end{bmatrix} = \begin{bmatrix}
  A - BK & BK \\
  0 & A - LC
\end{bmatrix} \begin{bmatrix}
  x(k) \\
  e(k)
\end{bmatrix} + \begin{bmatrix}
  B \\
  0
\end{bmatrix} v(k)
\]
Observer-based feedback control

Equations become

\[
x(k + 1) = Ax(k) + B(v(k) - Kx(k)) + B(Kx(k) - Kx'(k))
\]
\[
= (A - BK)x(k) + Bv(k) + BKe(k)
\]
\[
e(k + 1) = (A - LC)e(k)
\]

i.e.

\[
\begin{bmatrix}
    x(k + 1) \\
    e(k + 1)
\end{bmatrix}
= \begin{bmatrix}
    A - BK & BK \\
    0 & A - LC
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    e(k)
\end{bmatrix}
+ \begin{bmatrix}
    B \\
    0
\end{bmatrix}v(k)
\]

Block diagonal form \(\implies K, L\) can be designed independently