This paper contains 5 questions

Answer **three** out of **five** questions

An outline marking scheme is shown in brackets to the right of each question.

This examination contributes 100% of the marks for the module.

University approved calculators MAY be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

8 page examination paper.
Question 1.

(a) A linear discrete-time system is described by the state-space model

\[
\begin{align*}
    x(p + 1) &= \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(p) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(p) \\
    y(p) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(p)
\end{align*}
\]

Test this system for controllability and observability and determine its transfer-function. [14 marks]

(b) Obtain a difference equation description of the dynamics of the system of the previous part of this question. [6 marks]

(c) The Laplace transform of the unit step-response of a differential linear time-invariant system with output \( y(t) \) and input \( u(t) \) is expressed in partial fraction form as

\[
    Y(s) = \frac{1}{24s} - \frac{1}{8(s + 4)} + \frac{1}{12(s + 6)}
\]

Find the \( z \)-transform of the output when sampled with a sampling period of 2 secs. Repeat this computation for the case when the input is replaced by

\[
    u(p) = \begin{cases} 
        1, & 0 \leq p < 3 \\
        -7, & 3 \leq p < 6 \\
        7, & p \geq 6 
    \end{cases}
\]

You may make use of the following formula where all symbols have their normal meanings.

\[
    \mathcal{Z} \left( \frac{1}{s + c} \right) = \frac{z}{z - e^{-aT}}
\]

[13 marks]
Question 2.

(a) A discrete linear time-invariant system is described by the transfer-function

\[ G(z) = \frac{b}{z^4 - 3z^2 + 1} \]

where \( b \neq 0 \) is a real scalar. Construct for this system (i) the controllable canonical form and (ii) the observable canonical form. Design a state feedback control law for this system that places the closed-loop poles at \( z = -\frac{1}{2} \pm j\frac{1}{4}, \frac{1}{2}, -\frac{1}{2} \). [16 marks]

(b) Consider discrete linear time-invariant dynamics described by the state-space model

\[
\begin{align*}
x(p + 1) &= Ax(p) + Bu(p) \\
y(p) &= Cx(p)
\end{align*}
\]

It is required to design an observer for this system based on the following state equation where \( \hat{x}(p) \) denotes the state vector of the observer and \( L \) is the observer gain matrix

\[ \hat{x}(p + 1) = A\hat{x}(p) + Bu(p) + L(y(p) - C\hat{x}(p)) \]

Show that the error dynamics are described by

\[ e(p + 1) = x(p + 1) - \hat{x}(p + 1) = (A - LC)e(p) \]

Detail how the design of \( L \) is undertaken to ensure that \( e(p) \) converges to zero as \( p \to \infty \). [5 marks]

(c) Design an observer to implement the state feedback law of part (a) of this question with all observer poles at \( -\frac{1}{5} \). Give also the poles of the resulting controlled system. You may make use of the fact that for a variable, say, \( h \)

\[ (h + \frac{1}{5})^4 = h^4 + \frac{4}{5}h^3 + \frac{6}{25}h^2 + \frac{4}{125}h + \frac{1}{625} \]

[12 marks]

TURN OVER
Question 3.

(a) A unity negative feedback control scheme has system transfer-function \(G(s)\) and forward path controller \(K(s)\) and the controller is to be designed by direct digital control. The controller output is sampled with period \(T = 0.5\) sec, fed into a ZOH and the resulting output is applied to the system whose output is also sampled with the same period. The controller input is the difference between the sampled output and the sampled reference signal. A particular case of interest is when

\[
G(s) = \frac{s}{s^2 + 3s + 3}
\]

Confirm that \(G(s)\) can also be written as

\[
G(s) = \frac{\sqrt{3}s}{\frac{\sqrt{3}}{2}((s + \frac{3}{2})^2 + \frac{3}{4})}
\]

Show that the \(z\) transfer-function of \(G(s)\) connected to a ZOH and sampler is of the form

\[
G(z) = \alpha \frac{z - 1}{z^2 + \beta z + \gamma}
\]

and give the values of the constants \(\alpha, \beta\) and \(\gamma\).

You may make use of the following formula where \(c\) and \(d\) are real scalars and all other terms have their normal meanings.

\[
Z\left( \frac{c}{(s + d)^2 + c} \right) = \frac{ze^{-dT} \sin cT}{z^2 - 2ze^{-dT} \cos cT + e^{-2dT}}
\]

[10 marks]

(b) It is proposed to select the controller in the feedback control system of part (a) of this question as

\[
K(z) = \frac{1}{\alpha} \frac{z^2 + \beta z + \gamma}{z^2 + h_1 z + h_0}
\]
Show that the resulting closed-loop transfer-function is

\[ Q(z) = \frac{z - 1}{z^2 + (1 + h_1)z + h_0 - 1} \]

Comment on any implementation difficulties that may result from this choice of controller.

[10 marks]

(c) Consider the particular case when \( h_0 = 1 \) and \( h_1 = -1 \) in the controller in part (b) of this question and the reference signal is a step sequence of magnitude \( r \) applied at the origin. Give the poles and zeros of the closed loop system under this choice and find the resulting steady state value of the output. How does this value reflect the presence of a zero at \( s = 0 \) in \( G(s) \)?

[6 marks]

(d) Suppose that the reference signal for the control system of part (c) of this question is replaced by

\[ r(p) = \begin{cases} 
1, & p = 0 \\
0, & \text{for all } k > 0 
\end{cases} \]

Compute the system output \( y(p) \) for all \( p \geq 0 \).

[7 marks]
Question 4.

(a) A continuous linear time-invariant controllable and observable system is described by the state-space model

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\]

Compute the ZOH-discretization of this state-space model with sampling period \( T \). You may make use of the following matrix equation

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\]

[12 marks]

(b) Is the ZOH-discretization of the system of part (a) of this question controllable and observable? Do there exist values of the sampling period for which these properties are not present in the resulting discretization? If, for any system, there is a sampling period for which controllability and/or observability is lost, determine value of the strictest upper bound for the largest sampling period \( T \) that preserves the controllability and observability of the continuous linear time-invariant system.

[9 marks]

(c) Design a deadbeat observer for the system described by the state-space model.

\[
x(p + 1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 5 & 1 \end{bmatrix} x(p) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(p) \\
y(p) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(p)
\]

Explain what may happen to the performance if fast sampling is used in an implementation of a deadbeat state feedback control law using this form of observer and give one possible solution. [12 marks]
Question 5.

(a) Consider a discrete linear time invariant system described by

\[ x(p + 1) = Ax(p) + Bu(p) \]

where \( x(p) \) is of dimension \( n \times 1 \) and \( u(p) \) is of dimension \( m \times 1 \) and it is known that the pair \((A, B)\) is controllable. The control objective is to design the state feedback control law

\[ u(p) = -Kx(p) \]

to minimise the quadratic cost function

\[ J = \sum_{p=0}^{\infty} x^T(p)Qx(p) + u^T(p)Ru(p) \]

where the \( n \times n \) matrix \( Q \) is symmetric positive semi-definite and the \( m \times m \) matrix \( R \) is symmetric positive-definite.

The solution of this problem is

\[ u_{opt}(p) = -W^{-1}B^TPAx(p) \]

where

\[ W = B^TPB + R \]

and \( P \) is the solution of the algebraic Riccati equation

\[ A^TPA - P + Q - A^TPBW^{-1}B^TPA = 0 \]

Compute the optimal control law for the particular case when

\[ x(p + 1) = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{4}{5} \end{bmatrix} x(p) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(p) \]

\[ y(p) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(p) \]

and

\[ J = \sum_{k=0}^{\infty} (x_1^2 + x_1x_2 + x_2^2 + 0.1u^2) \]
Start by confirming that the solution of the algebraic Riccati equation in this case is

\[
P = \begin{bmatrix} 1.024 & 0.55 \\ 0.55 & 1.98 \end{bmatrix}
\]

Also compute the optimal cost when

\[ x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \]

and determine the stability properties of the resulting controlled system.

[20 marks]

(b) Consider the discrete linear time invariant system described by

\[
x_{p+1} = (1 - \beta)x_p + \alpha x_p u_p \\
y_{p+1} = (1 - \beta)y_p + \alpha x_p (1 - u_p)
\]

where \( \beta \in (0, 1) \) and \( \alpha > 0 \) are constants and \( 0 < \eta \leq u_p \leq \zeta < 1 \). Also \( x_0 > 0 \) and \( y_0 = 0 \). Formulate the dynamic programming problem to determine \( u_p \) such that \( x_p \) is maximal after \( N \) steps.

[13 marks]