SEMESTER 2 EXAMINATION 2015 - 2016

DIGITAL CONTROL SYSTEM DESIGN

DURATION 120 MINS (2 Hours)

This paper contains 5 questions

Answer three out of five questions

An outline marking scheme is shown in brackets to the right of each question.

This examination contributes 100% of the marks for the module.

University approved calculators MAY be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

16 page examination paper.

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Question 1.

(a) A linear discrete-time system is described by the state-space model

\[
x(p + 1) = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(p) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(p)
\]

\[
y(p) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(p)
\]

Test this system for controllability and observability and determine its transfer-function. [14 marks]

(b) Obtain a difference equation description of the dynamics of the system of the previous part of this question. [6 marks]

(c) The Laplace transform of the unit step-response of a differential linear time-invariant system with output \( y(t) \) and input \( u(t) \) is expressed in partial fraction form as

\[
Y(s) = \frac{1}{24s} - \frac{1}{8(s + 4)} + \frac{1}{12(s + 6)}
\]

Find the \( z \)-transform of the output when sampled with a sampling period of 2 secs. Repeat this computation for the case when the input is replaced by

\[
u(p) = \begin{cases} 
1, & 0 \leq p < 3 \\
-7, & 3 \leq p < 6, \\
7, & p \geq 6
\end{cases}
\]

You may make use of the following formula where all symbols have their normal meanings.

\[
\mathcal{Z} \left( \frac{1}{s + c} \right) = \frac{z}{z - e^{-at}}
\]

[13 marks]
Indicative Solution for Question 1.

(a) [14 marks]

Let

\[ A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = [0 \ 0 \ 1] \]

Controllability requires that

\[ \text{rank} \left( M_c \right) = \text{rank} \left[ B \ A\ B \ A^2 \ B \right] = 3 \]

where

\[ M_c = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \]

Also \( \det(M_c) \neq 0 \)

and hence this system is controllable.

Observability requires that

\[ \text{rank} \left( M_o \right) = \text{rank} \left[ C \ A \ C A \ A^2 \right] = 3 \]

where

\[ M_o = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

Also \( \det(M_o) \neq 0 \)

and hence this system is observable.

Transfer-function is

\[ G(z) = C(zI_3 - A)^{-1}B \]

\[ (zI_3 - A)^{-1} = \frac{1}{z^2(z+1) - 1} \begin{bmatrix} z + 1 & 1 & z \\ 1 & z(z+1) & 1 \\ z(z+1) & 1 & z + 1 \end{bmatrix} \]

Hence

\[ G(z) = \frac{z + 1}{z^2(z+1) - 1} \]

(b) [6 marks]

Write

\[ Y(z) = G(z)U(z) \]
or

\[(z^2(z + 1) - 1)Y(z) = (z + 1)U(z)\]

Using the definition of \(z\) as a forward shift operator and taking the inverse \(z\)-transform gives


This is one form of the solution. Another is to replace \(n\) by \(n - 3\) in this last equation, which will give the difference equation that would result from first dividing the transfer-function above and below by \(z^{-1}\). These are equivalent difference equation and hence either answer (if correct) would score full marks.

(c) [13 marks]

Using the supplied formula for the \(z\)-transform, gives for the first input signal

\[Y(z) = \frac{z}{24} - \frac{z}{8} - \frac{z}{12} + \frac{z}{z - e^{-8}} + \frac{z}{z - e^{-12}}\]

In the continuous-time domain the second input signal is equivalent to a unit step applied at \(t = 0\), a step function of \(-8\) applied at \(t = 3\) and then a step function of \(14\) applied at \(t = 6\). Let the step input applied at \(t = 0\) be denoted by \(u_s(t)\), i.e., \(u_s(t) = 1\), \(t > 0\). Hence

\[u(t) = u_s(t) - 8u_s(t - 3) + 14u_s(t - 6)\]

or in Laplace transform terms

\[U(s) = \frac{1 - 8e^{-3s} + 14e^{-6s}}{s}\]

Let \(Y'(z)\) be the \(z\)-transform of the output in this case. Then

\[Y'(z) = Y(z)(1 - 8z^{-3} + 7z^{-6})\]

(since the \(z\)-transform of \(e^{-bs}\) is \(z^{-b}\).)
Question 2.

(a) A discrete linear time-invariant system is described by the transfer-function
\[ G(z) = \frac{b}{z^4 - 3z^2 + 1} \]
where \( b \neq 0 \) is a real scalar. Construct for this system (i) the controllable canonical form and (ii) the observable canonical form. Design a state feedback control law for this system that places the closed-loop poles at \( z = -\frac{1}{2} \pm j\frac{1}{4}, \frac{1}{2}, -\frac{1}{2} \). [16 marks]

(b) Consider discrete linear time-invariant dynamics described by the state-space model
\[
\begin{align*}
    x(p + 1) &= Ax(p) + Bu(p) \\
    y(p) &= Cx(p)
\end{align*}
\]
It is required to design an observer for this system based on the following state equation where \( \hat{x}(p) \) denotes the state vector of the observer and \( L \) is the observer gain matrix
\[
\hat{x}(p + 1) = A\hat{x}(p) + Bu(p) + L(y(p) - C\hat{x}(p))
\]
Show that the error dynamics are described by
\[
e(p + 1) = x(p + 1) - \hat{x}(p + 1) = (A - LC)e(p)\]
Detail how the design of \( L \) is undertaken to ensure that \( e(p) \) converges to zero as \( p \to \infty \). [5 marks]

(c) Design an observer to implement the state feedback law of part (a) of this question with all observer poles at \( -\frac{1}{5} \). Give also the poles of the resulting controlled system. You may make use of the fact that for a variable, say, \( h \)
\[
(h + \frac{1}{5})^4 = h^4 + \frac{4}{5}h^3 + \frac{6}{25}h^2 + \frac{4}{125}h + \frac{1}{625}
\]
[12 marks]

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Indicative Solution for Question 2.

(a) [16 marks] Controllable companion form

\[ x(p + 1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 3 & -1 \end{bmatrix} x(p) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} u(p) \]

\[ y(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(p) \]

Observable companion form

\[ x(p + 1) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} x(p) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(p) \]

\[ y(p) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x(p) \]

Desired control system polynomial is

\[ \rho_c(z) = (z + \frac{1}{2} \pm j \frac{1}{4})(z - \frac{1}{2})(z + \frac{1}{2}) = z^4 + z^3 + \frac{1}{16} z^2 - \frac{1}{4} z - \frac{5}{64} \]

State feedback control law

\[ u(p) = -Kx(p) = -\begin{bmatrix} k_0 & k_1 & k_2 & k_3 \end{bmatrix} x(p) \]

Now we require

\[ \text{det}(zI_4 - A + BK) = \rho_c(z) \]

Using the formula given in the notes

\[ K = \begin{bmatrix} -\frac{69}{64} & \frac{49}{48} & -\frac{1}{4} & -\frac{69}{64} \end{bmatrix} \]

(b) [5 marks] The state equation for the error dynamics follows directly on substituting from the output equation for the plant model into the formula for \( \hat{x}(p+1) \) and then forming the error dynamics. To ensure that the error converges to zero requires that all eigenvalues of \( A - LC \) lie inside the unit circle in the complex plane. Given that the eigenvalues of a matrix are equal to those of its transpose, this is the pole placement problem for the system with state matrix \( A^T \), input matrix \( C^T \) and state feedback gain matrix \( -L \). This problem has a solution if and only if \( A^T, C^T \) is a controllable pair or, equivalently the system is observable.

(c) [12 marks] The requirement is to design

\[ L = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 \end{bmatrix}^T \]
such that (using the fact given in the question immediately gives the right-hand side below)
\[
\det (zI_4 - A + LC) = z^4 + \frac{4}{5}z^3 + \frac{6}{25}z^2 + \frac{4}{125}z + \frac{1}{625}
\]
Equating coefficients gives
\[
l_1 = -\frac{624}{625}, \quad l_2 = \frac{4}{125}, \quad l_3 = \frac{31}{25}, \quad l_4 = -\frac{1}{5}
\]
Poles of the resulting controlled system, are by the separation principle, the union of those assigned by the state feedback law and those of the error dynamics, i.e., the poles used in the design of L above.
Question 3.

(a) A unity negative feedback control scheme has system transfer-function $G(s)$ and forward path controller $K(s)$ and the controller is to be designed by direct digital control. The controller output is sampled with period $T = 0.5$ sec, fed into a ZOH and the resulting output is applied to the system whose output is also sampled with the same period. The controller input is the difference between the sampled output and the sampled reference signal. A particular case of interest is when

$$G(s) = \frac{s}{s^2 + 3s + 3}$$

Confirm that $G(s)$ can also be written as

$$G(s) = \frac{\sqrt{3}}{2} s \frac{\sqrt{3}}{2} \left(\frac{s + \frac{3}{2}}{2} + \frac{3}{4}\right)$$

Show that the $z$ transfer-function of $G(s)$ connected to a ZOH and sampler is of the form

$$G(z) = \alpha z^{-1} z^2 + \beta z + \gamma$$

and give the values of the constants $\alpha, \beta$ and $\gamma$.

You may make use of the following formula where $c$ and $d$ are real scalars and all other terms have their normal meanings.

$$\mathcal{Z}\left(\frac{c}{(s + d)^2 + c}\right) = \frac{ze^{-dT} \sin cT}{z^2 - 2ze^{-dT} \cos cT + e^{-2dT}}$$

[10 marks]

(b) It is proposed to select the controller in the feedback control system of part (a) of this question as

$$K(z) = \frac{1}{\alpha} \frac{z^2 + \beta z + \gamma}{z^2 + h_1 z + h_0}$$
Show that the resulting closed-loop transfer-function is

\[ Q(z) = \frac{z - 1}{z^2 + (1 + h_1)z + h_0 - 1} \]

Comment on any implementation difficulties that may result from this choice of controller.

[10 marks]

(c) Consider the particular case when \( h_0 = 1 \) and \( h_1 = -1 \) in the controller in part (b) of this question and the reference signal is a step sequence of magnitude \( r \) applied at the origin. Give the poles and zeros of the closed loop system under this choice and find the resulting steady state value of the output. How does this value reflect the presence of a zero at \( s = 0 \) in \( G(s) \)?

[6 marks]

(d) Suppose that the reference signal for the control system of part (c) of this question is replaced by

\[ r(p) = \begin{cases} 
1, & p = 0 \\ 
0, & \text{for all } k > 0
\end{cases} \]

Compute the system output \( y(p) \) for all \( p \geq 0 \).

[7 marks]
Indicative Solution for Question 3.

(a) [10 marks]

Let \( G(z) \) be the required transfer-function. Then

\[
G(z) = \frac{z - 1}{Z\left(\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} + \frac{3}{4}\right)}
\]

Using the supplied \( z \)-transform now gives \( G(z) \) of the form stated with

\[
\alpha = 0.377, \quad \beta = -2e^{-aT} \cos bT, \quad \gamma = e^{-2aT}, \quad a = \frac{3}{2}, \quad b = \frac{3}{4}, \quad T = \frac{1}{2}
\]

Recall that if a ZOH is connected in series with a transfer-function \( G(s) \), the \( z \)-transform is

\[
(1 - z^{-1})Z\left(\frac{G(s)}{s}\right)
\]

This explains why the \( s \) term in numerator of the given \( G(s) \) is cancelled and the given \( z \)-transform pair in the question is directly applicable.

(b) [10 marks]

\[
Q(z) = \frac{G(z)K(z)}{1 + G(z)K(z)} = \frac{z - 1}{z^2 + (1 + h_1)z + h_0 - 1}
\]

This is a pole-zero cancellation design and if there is uncertainty in controller implementation, i.e., in the coefficients, the pole-zero cancellation will not take place with possibly detrimental consequences.

(c) [6 marks] Setting \( h_0 = 1 \) and \( h_1 = -1 \) in the formula for \( Q(z) \) in the answer to the previous part gives

\[
Q(z) = \frac{z - 1}{z^2}
\]

Hence

\[
Y(z) = rz - 1 \frac{z - 1}{z} = \frac{r}{z}
\]

Hence \( y(p) = ry(p - 1) \). Hence \( y(0) = 0 \) for all \( p > 1 \) the output value is equal to \( r \). This is reflected by the presence of a zero at \( s = 0 \) in \( G(s) \) and a zero at \( z = 1 \) in \( G(z) \).

(d) [7 marks] In this case

\[
Y(z) = \frac{z - 1}{z^2} = \frac{1}{z} - \frac{1}{z^2}
\]

Hence

\[
y(0) = 0, \quad y(1) = 1, \quad y(2) = -1
\]

and

\[
y(p) = 0
\]

for \( p > 2 \).
Question 4.

(a) A continuous linear time-invariant controllable and observable system is described by the state-space model

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\]

Compute the ZOH-discretization of this state-space model with sampling period \(T\). You may make use of the following matrix equation

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\]

[12 marks]

(b) Is the ZOH-discretization of the system of part (a) of this question controllable and observable? Do there exist values of the sampling period for which these properties are not present in the resulting discretization? Also determine the numerical value of the strictest upper bound for the largest sampling period \(T\) that preserves the controllability and observability of the continuous linear time-invariant system of part (a) of this question.

[9 marks]

(c) Design a deadbeat observer for the system described by the state-space model.

\[
x(p + 1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 5 & 1 \end{bmatrix} x(p) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(p) \\
y(p) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(p)
\]

Explain what may happen to the performance if fast sampling is used in an implementation of a deadbeat state feedback control law using this form of observer and give one possible solution. [12 marks]
Indicative Solution for Question 4.

(a) [12 marks] Computing the eigenvalue-eigenvector decomposition of the state matrix, say $A$, gives

$$
\begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}
$$

The given matrix equation is an eigenvalue-eigenvector decomposition of the state matrix, i.e., the state matrix has eigenvalues $1$ and $-1$ and the matrix

$$
\begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}
$$

is the matrix of eigenvectors. The remaining matrix is the inverse of the eigenvector matrix.

Next use the formula

$$
\Phi(T) = e^{AT} = T^{-1} \begin{bmatrix}
e^t & 0 \\
0 & e^{-t}
\end{bmatrix} T
$$

and the associated input matrix is

$$
\Gamma = \int_0^T e^{A(T-\tau)}Bd\tau
$$

where $B$ denotes the input matrix of the continuous-time state-space model.

In terms of the answer to the next part, it is not necessary to work there formulas in any more detail.

(b) [9 marks] Controllability requires that

$$
\det \left[ \begin{array}{c}
\Gamma \\
\Phi \Gamma
\end{array} \right] \neq 0
$$

Observability requires that

$$
\det \left[ \begin{array}{c}
C \\
C \Phi
\end{array} \right] \neq 0
$$

where $C = [1 \quad 0]$.

The controllability and observability matrices will have entries that involve $e^T$ and $e^{-T}$ where the first of these functions is never negative and the second tends to zero as $T \to \infty$ only. Hence it can be conclude that in this case controllability and observability hold for all finite $T$.

If for a system there does exist a finite sampling period then it is possible that the choice of $T$ could result in a loss of controllability and/or observability. Controllability and observability of the continuous time systems is preserved under sampling if

$$
\frac{1}{T} > \frac{\text{im}\lambda_i}{\pi}
$$

where $\lambda_i$ is the $i$th eigenvalue of the continuous time system. Hence $T$ must be less than $\pi$.

(c) [12 marks]
Deadbeat observer — all eigenvalues of the matrix $A - LC$ must be at the origin of the complex plane, i.e.

$$\det(zI_3 - A + LC) = z^3$$

with

$$L = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T$$

Hence by equating coefficients

$$L = \begin{bmatrix} -1 & 5 & 1 \end{bmatrix}^T$$

Using a deadbeat controller with sampling period $t$ results in a setting time of $nT$ ($3T$ for the system considered). Hence the smaller the sampling period the shorter the settling time. This, in turn, can lead to large entries in the state feedback matrix and hence large input demands.

One way of dealing with this problem is to chose a larger sampling time corresponding to a larger sampling period. For a desired settling time $T_s$ chose

$$T = \frac{T_s}{n}$$
Question 5.

(a) Consider a discrete linear time invariant system described by

\[ x(p+1) = Ax(p) + Bu(p) \]

where \( x(p) \) is of dimension \( n \times 1 \) and \( u(p) \) is of dimension \( m \times 1 \) and it is known that the pair \( (A, B) \) is controllable. The control objective is to design the state feedback control law

\[ u(p) = -Kx(p) \]

to minimise the quadratic cost function

\[ J = \sum_{p=0}^{\infty} x^T(p)Qx(p) + u^T(p)Ru(p) \]

where the \( n \times n \) matrix \( Q \) is symmetric positive semi-definite and the \( m \times m \) matrix \( R \) is symmetric positive-definite.

The solution of this problem is

\[ u_{opt}(p) = -W^{-1}B^TPAx(p) \]

where

\[ W = B^TPB + R \]

and \( P \) is the solution of the algebraic Riccati equation

\[ A^TPA - P + Q - A^TPBW^{-1}B^TPA = 0 \]

Compute the optimal control law for the particular case when

\[ x(p+1) = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{4}{5} \end{bmatrix} x(p) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(p) \]
\[ y(p) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(p) \]

and

\[ J = \sum_{k=0}^{\infty} (x_1^2 + x_1x_2 + x_2^2 + 0.1u^2) \]
Start by confirming that the solution of the algebraic Riccati equation in this case is

\[ P = \begin{bmatrix} 1.024 & 0.55 \\ 0.55 & 1.98 \end{bmatrix} \]

Also compute the optimal cost when

\[ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \]

and determine the stability properties of the resulting controlled system.

[20 marks]

(b) Consider the discrete linear time invariant system described by

\[ x_{p+1} = (1 - \beta)x_p + \alpha x_p u_p \]
\[ y_{p+1} = (1 - \beta)y_p + \alpha x_p (1 - u_p) \]

where \( \beta \in (0, 1) \) and \( \alpha > 0 \) are constants and \( 0 < \eta \leq u_p \leq \zeta < 1 \). Also \( x_0 > 0 \) and \( y_0 = 0 \). Formulate the dynamic programming problem to determine \( u_p \) such that \( x_p \) is maximal after \( N \) steps.

[13 marks]
Indicative Solution for Question 5.

(a) [20 marks] \( J \) can be written in the required form for

\[
Q = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad R = 0.1
\]

The solution matrix of the algebraic Riccati equation \( P \) must be symmetric so take

\[
P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}
\]

Then

\[
\begin{align*}
0.25p_3 - p_1 + 1 - \frac{0.25p_3^2}{0.1 + p_3} &= 0 \\
0.4p_3 - 0.5p_2 + 0.5 - \frac{0.5p_2p_3 + 0.4p_3^2}{0.1 + p_1} &= 0 \\
p_1 + 1.6p_2 - 0.36p_3 + 1 - \frac{p_2^2 + 1.6p_2p_3 + 0.64p_3^2}{0.1 + p_2} &= 0
\end{align*}
\]

Now substitute the given entries in \( P \).

Optimal cost is

\[
J_o = x^T(0)Px(0) = 4.1075
\]

The optimal control problem solution is a stabilizing state feedback law.

(b) [13 marks]

The material on which this part of the question is based has not been covered this year.