ELEC3215
FLUIDS AND
MECHANICAL MATERIALS

Hydrostatics I: Pressure
Fluids at rest: hydrostatics

As in any general system, a static system is at rest and time derivatives are zero. As mentioned previously, for a fluid, there are no shearing forces and all forces act perpendicularly.

At a solid interface, the force must act on the boundary at right angles.
Fluids at rest: hydrostatics

If we have a curved surface or interface, the force must act on the boundary everywhere at right angles. If a piecewise representation of the surface is considered then the force on each element is perpendicular.
Fluids at rest: hydrostatics

By extension, for the fluid itself, if we draw any plane through the continuum of the fluid, then all internal forces exerted by the element of fluid to one side of plane must act perpendicularly on the interface with the fluid on the other side of the plane.
Fluids at rest: hydrostatics

**Moving fluids:** Shear stresses are only generated (from Newton) when there is relative motion between fluid volume elements. If each element of the fluid is moving but all are stationary relative to each other, then the principles of static fluids still applies.

When fluid systems are analysed, it is normal to consider volume elements as free bodies defined by solid or imaginary boundaries, and then consider the forces acting on it. As we will see later, this leads to per unit volume equations and the consideration of forces moving with the fluid.

If the fluid element is static (in equilibrium), the sum of the component forces acting in any direction must be zero. An imbalance in this case would lead to translational motion. Similarly, the sum of the moments of the forces about any point must be zero – this would lead to rotation in the event of an imbalance. This would normally be tested along three orthogonal axes and in three orthogonal planes, although the selection of these is arbitrary.
Equilibrium: as we will see later, the consideration of equilibrium and stability for static systems is an important engineering area e.g. will an apparently stable floating ship right itself following a roll? There are three types of equilibrium:

**Stable equilibrium:** a small displacement generates a restoring force.

**Unstable equilibrium:** a small displacement produces a moment which increases the displacement from the equilibrium position.

**Neutral equilibrium:** the object will move under the action of the force and will stop in equilibrium again in any position.
Pressure

A fluid will exert a force normal to any boundary or any imaginary plane drawn through the fluid. Since the system we are interested in may be infinite (flow through a pipe) and most likely the force will vary with position in the fluid, it is easier to describe the forces in terms of *pressure*:

\[
\text{Pressure} = \frac{\text{Force}}{\text{Area force exerted on}} \quad \text{or} \quad p = \frac{F}{A}, \quad \text{units} = \text{Nm}^{-2}
\]

For positional variation, we consider the force element \( \delta F \) normal to area element \( \delta A \):

Mean pressure, \( p = \frac{\delta F}{\delta A} \)

Pressure at a point, \( p = \lim_{\delta A \to 0} \frac{\delta F}{\delta A} = \frac{dF}{dA} \)
Pascal’s Law for pressure at a point

Consider a triangular prism volume element as shown:
Pascal’s Law for pressure at a point

Since the fluid is at rest, there are no shearing forces on the surfaces of the volume element and the element will not be accelerating. The sum of the forces in any direction must be zero:

\[ x\text{-direction} \quad \text{force due to } p_x = p_x A_{ABFE} \]
\[ = p_x \delta y \delta z \]

\[ x\text{-component of force due to } p_s = -p_s A_{ABCD} \sin \theta \]
\[ = -p_s \delta s \delta z \frac{\delta y}{\delta s} \quad \text{since } \sin \theta = \frac{\delta y}{\delta s} \]
\[ = -p_s \delta y \delta z \]

As \( p_y \) is orthogonal, element is in equilibrium if
\[ p_x \delta y \delta z - p_s \delta y \delta z = 0 \]
\[ \therefore p_x = p_s \]
Pascal’s Law for pressure at a point

_y-direction_  

force due to \( p_y = p_y A_{CDEF} \)

\[ = p_y \delta x \delta z \]

_y-component of force due to \( p_s = -p_s A_{ABCD} \cos \theta \)

\[ = -p_s \delta s \delta z \frac{\delta x}{\delta s} \quad \text{since} \quad \cos \theta = \frac{\delta x}{\delta s} \]

\[ = -p_s \delta x \delta z \]

In addition:  

Weight of element = \(-\) specific weight \(\times\) volume

\[ = -\rho g \times \frac{1}{2} \delta x \delta y \delta z \]

As \( p_x \) is orthogonal,

\[ p_y \delta x \delta z - p_s \delta x \delta z - \rho g \times \frac{1}{2} \delta x \delta y \delta z = 0 \]

Since \( \delta x, \delta y \) and \( \delta z \) are small, then \( \delta x \delta y \delta z \) is very small and \( p_y = p_s \)
Pascal’s Law for pressure at a point

Combining these two results, we obtain:

\[ p_x = p_y = p_z \]

Since the plane can be at any angle, and we can generalise to include the \( z \)-direction, we obtain the result that in a static fluid, the pressure is the same in all directions.

This is **Pascal’s Law for pressure at a point**

If the fluid is moving, with shear stresses acting on the elements, the pressure is the mean of the normal forces per unit area (stresses) on three mutually perpendicular planes. If the normal stresses are much larger than the shear stresses (usually the case) then we can assume that Pascal’s law still applied.
Variation of pressure vertically

To consider the effect of gravity on pressure, we examine the cylindrical volume element shown of a fluid of density $\rho$.

The forces acting are:

- Force due to $p_1$ on area $A$ acting upwards
  \[ = p_1 A \]

- Force due to $p_2$ on area $A$ acting downwards
  \[ = -p_2 A \]

- Force due to the weight of the element:
  \[ = -mg = -\text{mass density} \times \text{volume} \times g = -\rho A g (z_2 - z_1) \]
Variation of pressure vertically

Since there are no shear forces (fluid is at rest), there are no vertical forces acting on the sides of the element due to the surrounding fluid. Assuming that upwards is positive, the sum of the forces must be zero:

\[ p_1 A - p_2 A - \rho A g (z_2 - z_1) = 0 \]

Giving that

\[ p_1 - p_2 = \rho g (z_2 - z_1) \]

with the result that in a fluid experiencing gravitational attraction, pressure decreases with increasing height.
Equality of pressure at same level

If P and Q are two points at the same level in a fluid at rest, then the cylinder of fluid shown is in equilibrium. The horizontal forces acting on the element at P and Q are

\[ p_1 A \quad \text{and} \quad p_2 A \]

For static equilibrium,

\[ p_1 A = p_2 A \]

\[ p_1 = p_2 \]

Giving that the pressure at any two points at the same level must be equal. Formally, this should be written:

\[ \frac{\partial p}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p}{\partial y} = 0 \]
Equality of pressure at same level

The pressure equilibrium is maintained even without a direct path between the two points. If we have two vessels connected by a horizontal pipe as shown, we have

\[ p_R = p_S \]

and also that

\[ p_R = p_P + \rho z g \quad \text{and} \quad p_S = p_Q + \rho z g \]

Substituting gives

\[ p_P + \rho z g = p_Q + \rho z g \]
\[ p_P = p_Q \]
Point to point pressure variation

The general equation for the variation in pressure due to gravity:

\[ \text{force} \quad p_1 A \]

\[ \text{area} \ A \quad \text{pressure} \ p_1 \]

\[ \text{force} \quad (p_1 + \delta p) A \]

\[ \text{area} \ A \quad \text{pressure} \ p_1 + \delta p \]

\[ \delta s \]

\[ z + \delta z \]

\[ \delta \theta \]
Point to point pressure variation

We have a cylindrical fluid element between point P (pressure $p$) and point Q (pressure $p + \delta p$). The element is inclined at an angle of $\theta$ to the vertical and the heights of P and Q above an arbitrary reference level are $z$ and $z + \delta z$. The forces acting on the element are:

- pressure acting along axis of element on surface at P: $pA$
- pressure acting along axis of element on surface at Q: $(p + \delta p)A$
- weight of the element acting vertically downwards: $mg = \rho \times A \delta s \times g$

- There are also forces acting on the sides of the element from to the surrounding fluid. These forces are perpendicular to the axis PQ since the fluid is at rest and they are therefore normal to the sides of the element.
Point to point pressure variation

For the element to be in equilibrium, the sum of the forces in any direction must be zero. Along the axis PQ,

$$pA - (p + \delta p)A - \rho g A \delta s \cos \theta = 0$$

$$\delta p = -\rho g \delta s \cos \theta$$

In differential form as $\delta s \to 0$, we obtain the general 3D result:

$$\frac{dp}{ds} = -\rho g \cos \theta = -w \cos \theta$$
Point to point pressure variation

Specific case PQ horizontal, $\theta = 90^\circ$

$$\frac{dp}{ds}_{\theta=90^\circ} = \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$$

giving, as before that pressure everywhere on a horizontal plane is constant.

Specific case PQ vertical, $\theta = 0^\circ$

$$\frac{dp}{ds}_{\theta=0^\circ} = \frac{\partial p}{\partial z} = -\rho g \quad \text{and since} \quad \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$$

we obtain the same result as before, that:

$$\frac{dp}{dz} = -\rho g = -w$$
Point to point pressure variation

Additional points. Consider any two horizontal planes with vertical spacing $z$:

- Pressure everywhere on lower plane = $p$
- Pressure everywhere on upper plane = $p + z \frac{\partial p}{\partial z}$

Difference of pressure = $z \frac{\partial p}{\partial z}$

Since in the horizontal planes, the pressure must be constant, we have the result that $\frac{\partial p}{\partial z}$ also cannot vary horizontally. Therefore,

$$\frac{dp}{dz} = -\rho g$$

implies that the density must be constant over any horizontal plane.
Summary

The conditions for equilibrium under gravity are:

1. The pressure at all points in any horizontal plane must be the same
2. The density at all points in any horizontal plane must be the same
3. The change of pressure with elevation is given by \( \frac{dp}{dz} = -\rho g \)

The variation with pressure with elevation is found by integrating this expression:

\[
dp = -\int \rho g \, dz \quad \text{or} \quad p_2 - p_1 = -\int_{z_1}^{z_2} \rho g \, dz = -\int_{z_1}^{z_2} wdz
\]

which requires knowledge of the relationship between \( p \) and \( \rho \).
The density can be assumed to be constant for most liquids and for gases if the pressure differences are very small

\[ p = -\int \rho g \, dz \]

\[ = -\rho g \int dz \]

\[ = -\rho gz + \text{constant} \]

or for any two points at heights \( z_1 \) and \( z_2 \) above a reference

\[ p_2 - p_1 = -\rho g (z_2 - z_1) = -w(z_2 - z_1) \]
The relationship between pressure, density and temperature for an ideal (perfect) gas is given by

\[ \frac{p}{\rho} = RT \]

if we assume an isothermal system, then \( \rho = \frac{p}{RT} \) can be used to integrate

\[ \frac{dp}{dz} = -\rho g = -\frac{pg}{RT} \implies \frac{dp}{p} = -\frac{g}{RT} dz \]

integrating both sides from \( z_1 (p_1) \) to \( z_2 (p_2) \)

\[ \ln \left( \frac{p_2}{p_1} \right) = -\frac{g}{RT} (z_2 - z_1) \]

\[ \frac{p_2}{p_1} = e^{-\frac{g}{RT} (z_2 - z_1)} \]
Examples

This type of model for a gas will be revisited later in the notes, particularly in the sections on Thermodynamics. For the more complicated adiabatic case, see later or Douglas section 2.9 as an additional example.
Absolute vs Gauge ("Gage") pressure

An aside on terminology:

When calculation are made involving pressure in a fluid, measurements are made relative to a reference pressure. There are two commonly used terms for the measured pressure:

**GAUGE pressure**: pressure relative to atmospheric pressure.

**ABSOLUTE pressure**: pressure relative to a perfect vacuum.

The relationship between the two measurements is simply:

\[ p_{abs} = p_{gauge} + p_{atm} \]

Typical values (+notation) \[ p_{abs} = 150kPa(gauge) + 98kPa(abs) = 248kPa(abs) \]
Pressure and Head

From before, the relationship \( \frac{dp}{dz} = -\rho g \) can be integrated to give

\[
p = -\rho g z + \text{constant}
\]

In a liquid, the pressure \( p \) at any depth measured from the surface i.e. \( z = -h \) is

\[
p = \rho gh + \text{constant}
\]

\[
= \rho gh + p_{\text{atm}}
\]

since the pressure at the surface of a free liquid is usually atmospheric pressure. If \( p \) is measured as a gauge pressure, then this is

\[
p = \rho gh
\]
Pressure and Head

This is equivalent to saying that (assuming \( g \) is constant), that the gauge pressure at depth can be define by stating the height \( h \) of an equivalent column of the same fluid which would be necessary to produce the same pressure.

This height \( h \) is called the HEAD, with the equation:

\[
h = \frac{p}{\rho g}
\]

Example values:  
- water  \( \rho = 10^3 \text{ kgm}^{-3} \)  
- atmospheric pressure  \( p_{\text{atm}} = 10^5 \text{ Nm}^{-2} \)  
- head  
  \[
h = \frac{10^5}{10^3 \times 9.81} = 10.19 \text{ m}
\]
The hydrostatic paradox

Presented differently: Mott (p61)

Fluid is the same in all containers

Pressure at the bottom is the same for all containers regardless of area
The hydrostatic paradox

Presented differently: Douglas (p45)

Fluid is the same in all containers, bottom surface area is the same

\[
p = \rho gh \quad \text{force on base} = pA = \rho ghA
\]
The hydrostatic paradox

Water towers: sited with sufficient elevation (height) to maintain high pressure in the local system without requiring pumps.
Measuring pressure: manometer

The relationship between pressure and head is used to measure the pressure in a manometer or liquid gauge, also known as a piezometer. In its simplest form it is a tube inserted into a pipe under pressure.

\[ p_A = \rho gh_1 \]

\[ p_B = \rho gh_2 \]
Measuring pressure: manometer
Measuring pressure: manometer

A U-tube gauge can be used to measure the pressure of liquids or gases. The bottom of the u-tube is filled with the manometric liquid which must be:

- Immiscible with the fluid being measured
- Have greater density $\rho_{\text{man}} > \rho$
Measuring pressure: manometer

pressure equilibrium

\[ \Rightarrow p_B = p_C \]

On the pipe side

\[ p_B = p_A + \text{pressure due to height } h_1 \text{ of fluid } \rho \]

\[ = p_A + \rho g h_1 \]

On the tube side

\[ p_C = p_D + \text{pressure due to height } h_2 \text{ of measurement fluid } \rho_{\text{man}} \]

\[ \text{atmospheric pressure (absolute)} \]

\[ 0 \quad \text{(gauge pressure)} \]

\[ p_C = 0 + \rho_{\text{man}} g h_2 \]

equating

\[ p_B = p_C \]

\[ p_A + \rho g h_1 = \rho_{\text{man}} g h_2 \]

\[ p_A = (\rho_{\text{man}} h_2 - \rho h_1) g \]
Measuring pressure: manometer
Measuring pressure: manometer

A U-tube gauge can be used to measure the pressure difference between two points in a pipeline:

\[ p_C = p_A + \rho g h_A \]

\[ p_D = p_B + \rho g (h_B - h) + \rho_{\text{man}} gh \]

\[ p_A + \rho g h_A = p_B + \rho g (h_B - h) + \rho_{\text{man}} gh \]

pressure difference \[ p_A - p_B = \rho g (h_B - h_A) + (\rho_{\text{man}} - \rho) gh \]
Measuring pressure: manometer

In the preceding examples, if the fluid being measured is a gas, then the density $\rho$ can be neglected in comparison to the measurement fluid density $\rho_{\text{man}}$. The expressions for the pressure then simplify to just measuring the difference in height between the two menisci. Otherwise heights relative to reference heights must also be included as measurements (1 for the first u-tube and 2 for the second).

Problems and disadvantages:

- Need to ensure that the tube connection to the measurement site is perfectly horizontal or the height measurements are incorrect

- Flush joints and seals are necessary or local pressure gradients are generated

- The movement of the liquid in both arms of the tube is necessary
Measuring pressure: manometer

One solution to the last problem is to alter the displacement cross-section in one of the two arms:

By making the diameter on one side much greater than the other, it is possible to reduce the movement on that side so that it is only necessary to measure the liquid movement on the narrower side.
Measuring pressure: manometer

If a pressure difference $p_1 - p_2$ is applied between the two sides of the manometer and the density of the gas being measured is negligible, for a starting level of AB in the two fluids:

- Volume of liquid required to produce rise of $h$ in RH tube: $h \left( \frac{\pi}{4} \right)^2 d^2$

- Fall in level of LH: $h \left( \frac{\pi}{4} \right)^2 d^2 = h \frac{d^2}{D^2}$

Pressure difference is given by total rise in level:

$$p_1 - p_2 = \rho g \left( h + h \frac{d^2}{D^2} \right) = \rho g h \left( 1 + \frac{d^2}{D^2} \right)$$

or, if the larger diameter is very much greater: $p_1 - p_2 = \rho g h$
If the pressure difference is very small, then the measurement leg of the u-tube can be inclined at an angle.

Measurement distance $x = \frac{h}{\sin \theta}$ is much greater than $h$ and:

$$p_1 - p_2 = \rho gh = \rho g x \sin \theta$$

An angle of 10° gives an increase in measurement length (accuracy) of 5.75.
Measuring pressure: manometer

Inverted u-tube: typically used for measuring pressure differences of liquids with gas as the measurement fluid.

\[ p_C = p_A + \rho g h_A + \rho_{man} g h \]

Pressure difference:

\[ p_A - p_B = \rho g (h_B - h_A) + (\rho - \rho_{man}) g h \]

If A and B at same level:

\[ p_A - p_B = (\rho - \rho_{man}) g h \]

Neglecting gas density:

\[ p_A - p_B = \rho g h \]
Measuring pressure: manometer

The manometer is a useful type of pressure gauge but has limitations:

- It can be adapted to measure small pressure differences but not large ones.

- Large pressure differences require serial application and a high density measurement fluid such as mercury.

- It does not require calibration but does require knowledge of temperature as density depends heavily.

- Surface tension and the shape of menisci is very important and therefore method is very sensitive to this. Likewise bubbles trapped in tubes has a large effect.

- Pressure fluctuations (common) produce oscillation of menisci and affect measurement accuracy.
Measuring pressure: other

Mott sections 3.8-3.10

A barometer is a device used for measuring atmospheric pressure. It contains a small volume of perfect (nearly vacuum) in an inverted tube submerged in mercury.
Final notes: relative equilibrium

If a fluid is at rest or moving at a constant speed, the pressure distribution is not affected by the motion of the container.

If the container is undergoing continual acceleration, this is transmitted into the fluid but since the fluid and the individual particles of the fluid do not move relative to the container, there are no shear stresses and the fluid pressures again are everywhere normal to the surfaces they act on.

The fluid is then said to be in RELATIVE EQUILIBRIUM.
Final notes: relative equilibrium

A liquid in a tank under acceleration is shown:

A fluid particle of mass $m$ at point A will have the same acceleration as the tank and be subjected to an accelerating force $F$:

$$F = ma$$
In addition, $F$ is the resultant of the fluid pressure force $F_p$ acting normally to the surface at A and the weight of the particle $mg$. Therefore:

$$F = mg \tan \theta$$

Equating the two expressions gives:

$$\tan \theta = \frac{a}{g}$$

which is the same for all points on the surface. The free surface must therefore be a plane inclined at an angle of $\theta$ to the horizontal.

The acceleration is horizontal and the vertical forces are not changed, then the pressure at any depth is $\rho gh$ and planes of equal pressure lie parallel to the free surface (dashed line in the figure).
Final notes: relative equilibrium

If the acceleration is vertical, the free surface remains horizontal.

Consider the prism of cross-sectional area \( A \) subject to upwards acceleration \( a \) which has a bottom surface at depth \( h \) below the surface experiencing a pressure of \( p \).

\[
\text{upwards accelerating force } F = \text{force due to } p - \text{weight of prism}\\
= pA - \rho ghA
\]

Newton:
\[
F = \text{mass of prism} \times \text{acceleration} = \rho hA \times a
\]

Therefore:
\[
pA - \rho ghA = \rho hA \times a \Rightarrow p = \rho gh \left( 1 + \frac{a}{g} \right)
\]

Douglas sec 2.19 for full general expression