ELEC3215
FLUIDS AND
MECHANICAL MATERIALS

Hydrostatics II: Forces
Pressure in fluid and forces

The fact that pressure, a scalar quantity, at depth is constant for any given depth and equal in all directions, leads to the conclusion that submerged surfaces and bodies are subject to an applied force, a vector quantity (magnitude *and* direction) which acts on the surface or through the body.

This section deals with static forces on submerged surfaces and the resulting mechanical effects of buoyancy and stability. Knowledge of static forces and the relationship with pressure is also important for the design and structure of containers and pipes etc...

There are two cases:

- a typical gas, where pressure is uniform throughout the fluid due to low specific weight and the force is therefore unaffected by altitude variations in pressure.
- a typical liquid where pressure increases with depth and the force calculation is more involved.
Action of pressure on surface

The pressure is the force per unit area, so for an arbitrary surface, flat or curved, or any boundary, the total force can be determined by considering the pressure acting across small volume elements of the surface.

In a body of fluid, the force may vary from point to point and the resulting force on the area element varies as well. In a hydrostatic system, the direction of the force is perpendicular to the surface at the volume element.

The total force is then the sum of the forces on the area elements.
Action of pressure on surface

The fluid pressure acting on the elements of a plane surface produces forces which are all parallel. In this case, the force on the surface can be represented by a single force called the Resultant Force which acts perpendicular to the plane at a point called the Centre Of Pressure.

The resultant force, $F_R$, is the sum of forces on all elements of area making up the surface:

$$F_R = p_1 \delta A_1 + p_2 \delta A_2 + \ldots + p_n \delta A_n = \sum_{i=1}^{n} p_i \delta A_i$$

If the surface is curved, the forces are not parallel and the total force is found by resolution or by polygon methods but will always be less than for a plane surface of the same area. Extreme case: cylinder.
Pressure on plane surfaces

For a uniform pressure there are two examples:

- A horizontal surface in a liquid
- Any surface in a sufficiently light gas

In both cases, the resultant force is the same:

\[ F_R = pA \]

where \( A \) is the total area of the surface, acting through the same centre of pressure, in this case the centroid of the surface. In the liquid, the force acts vertically (upwards or downwards depending on situation) and in the gas acting in the direction perpendicular to the orientation of the plane.
Pressure on plane surfaces

For an arbitrary plane surface immersed in a fluid, we must consider the variation of pressure with depth.
Pressure on plane surfaces

The surface has an area $A$ and is denoted PQ in linear section in the direction in which it is inclined at an angle of $\theta$ to the free surface of the liquid. We consider on side of the surface only (for obvious reasons) as if it were the bottom of a tank for example.

- There is a force due to the pressure $p$ acting on each area element $\delta A$.
- The pressure $p$ is a function of depth which for the element is denoted $y$.
- We assume that the pressure at the free surface is zero.

**QUESTION:** if we consider a typical example of this to be the pressure on a bucket, why work in gauge pressure?
Pressure on plane surfaces

Pressure at free surface is zero and working with depth as a measure i.e. \( y \) points downwards:

Force on area element \( \delta A \) is \( p \delta A = \rho g y \delta A \)

We sum all the forces on all area elements over the whole surface, since they are all parallel and perpendicular to PQ to give the resultant force on the surface:

\[
F_R = \sum \rho g y \delta A
\]

If we assume that \( \rho \) and \( g \) are constant everywhere:

\[
F_R = \rho g \sum y \delta A
\]
Pressure on plane surfaces

The quantity $\sum y\delta A$ is the first moment of area under the surface PQ about the free surface of the area and is equal to $Ay'$ where $A$ is the total area of the submerged surface and $y'$ is the depth of the centroid $G$. Substituting:

Resultant force is

$$F_R = \rho g Ay'$$

This resultant force will act perpendicular to the immersed surface at the centre of pressure $C$ at depth $D$ below the free surface. The moment if $F_R$ about any point will therefore be equal to the sum of the moments of the forces on all elements $\delta A$ about the same point. As a result, if the plane of the immersed surface PQ cuts the surface at O:

Moment of $F_R$ about O = sum of moments of forces on elements $\delta A$ about O

Force on any small element

$$= \rho g \delta Ay = \rho gs \sin \theta \times \delta A$$

$y = s \sin \theta$
Pressure on plane surfaces

Moment of force on element about O = \( \rho g s \sin \theta \times \delta A \times s \)

\[ = \rho g \sin \theta \times \delta A \times s^2 \]

Since \( \rho, g \) and \( \theta \) are the same for all elements,

Sum of the moments of all forces about O = \( \rho g \sin \theta \sum s^2 \delta A \)

In addition, \( F_R = \rho g Ay' \)

Moment of \( F_R \) about O = \( \rho g Ay' \times |OC| = \rho g Ay' \left( \frac{D}{\sin \theta} \right) \)

We equate these two derivations.
Pressure on plane surfaces

Equating:
\[ \rho g Ay' \left( \frac{D}{\sin \theta} \right) = \rho g \sin \theta \sum s^2 \delta A \]

\[ Ay' D = \sin^2 \theta \sum s^2 \delta A \]

\[ D = \sin^2 \theta \frac{\sum s^2 \delta A}{Ay'} \]

The term \( \sum s^2 \delta A \) is the second moment of area of the immersed surface about an axis in the free surface through O, which is also

\[ I_o = Ak_o^2 \]

where \( k_o \) is the radius of gyration of the immersed surface of the immersed surface about O.
Pressure on plane surfaces

Substituting again:  
\[ D = \sin^2 \theta \sum s^2 \delta A \frac{1}{Ay'} = \sin^2 \theta \frac{I_o}{Ay'} = \sin^2 \theta \frac{k_o^2}{y'} \]

The values of \( I_o \) and \( k_o \) can be found if the second moment of area of the immersed surface \( I_G \) about an axis through its centroid \( G \) parallel to the surface is known (or can be calculated). Using the parallel axis rule:

\[ Ak_o^2 = Ak_G^2 + A \left( \frac{y'}{\sin \theta} \right)^2 \quad \text{or} \quad k_o^2 = k_G^2 + \left( \frac{y'}{\sin \theta} \right)^2 \]

we get:

\[ D = \sin^2 \theta \frac{k_o^2}{y'} = \sin^2 \theta \left( \frac{k_G^2 + \left( \frac{y'}{\sin \theta} \right)^2}{y'} \right) \]

\[ = \sin^2 \theta \frac{k_G^2}{y'} + y' \]
Pressure on plane surfaces

The equation: \( D = \sin^2 \theta \left( \frac{k_G^2}{y'} \right) + y' \) says that the centre of pressure will always be below the centroid unless the plane is lying horizontally. However, as overall depth of the plane increases, the centre of pressure will move close to the centroid as the ratio of the difference in pressure (top to bottom of plane) to the average pressure decreases, i.e. the pressure difference is more uniform.

Finally, the lateral position of the centre of pressure can be found by taking moments about the line OG

\[
F_R \times d = \text{sum of moments of forces on small elements about OG}
\]

\[
= \sum \rho g y \delta Ax
\]

with \( F_R = \rho g Ay' \) then \( d = \frac{\sum \rho g y \delta Ax}{\rho g Ay'} = \frac{\sum y \delta Ax}{Ay'} \)

If the plane is symmetrical about the vertical plane through the centroid, then the moments are also symmetrical and balanced and \( d = 0 \).
Examples and methods

We will now look at a couple of example calculations and some information from Douglas and Mott.

Calculations can be performed using the moments to determine the resultant force or using pressure diagrams (Douglas section 3.4).

More examples can be found in Mott (pp 105 – 118) and Douglas (pp85 – 86)
# Shapes and centroids

<table>
<thead>
<tr>
<th>shape</th>
<th>details</th>
<th>Area $A$</th>
<th>$2^{nd}$ moment $I_G$ about axis PQ</th>
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</table>
From Douglas p65. A trapezoidal door in the vertical wall of a tank is hinged along its top edge as shown.
Trapezoidal door problem

Calculate:

• The resultant force $F_R$.
• The location of the centre of pressure C.
• The moment about the hinge necessary to hold the door closed.

The resultant force is given by $F_R = \rho g A y'$

The area of the plate is $A_{plate} = A_{trapezium} = \frac{length_{top} + length_{bottom}}{2} \times height$

$$= \frac{2.7 + 1.2}{2} \times 1.5 = 2.925 \text{ m}^2$$

To find the position of the centroid, take the moments of the area around $Q_1Q_2$ with distance $QG = y$. 
Trapezoidal door problem

The depth \( y \) can be found by comparison of the sum and the total:

\[
A \times y = \text{moment of area } Q_1Q_1'P_1 + \text{moment of area } Q_2Q_2'P_2 + \text{moment of area } P_1Q_1'Q_2'P_2
\]

\[
= 2 \times A_\Delta \times y_\Delta + A_\Box \times y_\Box
\]

\[
= 2 \times \left( \frac{b_\Delta}{2} \times h \right) \times y_\Delta + b_\Box \times h \times y_\Box
\]

\[
= 2 \times \left( \frac{0.75}{2} \times 1.5 \right) \times 0.5 + (1.2 \times 1.5) \times 0.75
\]

\[2.925 \, y = 0.5625 + 1.35 = 1.9125\]

\[\therefore y = 0.654 \, m\]

Depth to centroid of trapezium \( y' = y + ?? = 0.654 + 1.1 = 1.754 \, m^2\)
Trapezoidal door problem

The resultant force is then \( F_R = \rho g A y' = 10^3 \times 9.81 \times 2.925 \times 1.754 = 50.33 \text{kN} \)

Depth to centre of pressure
\[
D = \sin^2 \theta \frac{I_o}{A y'} \quad I_o = \sum s^2 \delta A
\]

From parallel axis, we have
\[
A k_o^2 = A k_G^2 + A \left( \frac{y'}{\sin \theta} \right)^2 \quad \text{and} \quad I_o = I_G + A \left( \frac{y'}{\sin \theta} \right)^2
\]

We also know that
\[
A_{\square} = bh \quad I_{G,\square} = \frac{bh^3}{12}
\]
\[
A_{\triangle} = \frac{bh}{2} \quad I_{G,\triangle} = \frac{bh^3}{36}
\]
\[\theta = 90^\circ \quad \text{and} \quad \therefore \sin \theta = 1\]
Calculating the second moment of area:

\[
I_o = 2 \left[ I_G + A \left( \frac{y'}{\sin \theta} \right)^2 \right] + \left[ I_G + A \left( \frac{y'}{\sin \theta} \right)^2 \right]_{\Delta}
\]

\[
= 2 \left[ \frac{b_{\Delta}h^3}{36} + \frac{b_{\Delta}h}{2} \left( y'_{\Delta} \right)^2 \right] + \left[ \frac{b_{\Delta}h^3}{12} + b_{\Delta}h \left( y'_{\Delta} \right)^2 \right]
\]

\[
= 2 \left[ \frac{b_{\Delta}h^3}{36} + \frac{b_{\Delta}h}{2} \left( \frac{0.75 \times 1.5^3}{2} + (1.1 + 0.5)^2 \right) + \left[ \frac{b_{\Delta}h^3}{12} + b_{\Delta}h \left( \frac{0.75 \times 1.5^3 + 1.2 \times 1.5^3}{12} + 1.2 \times 1.5 \times (1.1 + 0.75)^2 \right) \right]
\]

\[
= 2(0.0703125 + 0.5625 \times 1.6^2) + 0.3375 + 0.8 \times 1.85^2
\]

\[
= 3.020625 + 0.3375 + 6.1605
\]

\[
= 9.518625 \text{ m}^4
\]
Trapezoidal door problem

Depth to centre of pressure

\[ D = 1 \times \frac{9.5186}{2.925 \times 1.754} = 1.8553 \text{ m} \]

The moment of the resultant force about the hinge:

\[ \text{moment } F_R, Q_1Q_2 = F_R \times Q_1Q_2 = 50.33(1.8553 - 1.1) = 38.01 \text{ kN m} \]

Extra question: what force is a clasp at the bottom edge of the plate required to exert to hold the door closed?
Rectangular window at an angle

From Mott page 90. A rectangular window in a tank, set at an angle ($\theta = 60^\circ$).

The liquid is oil with specific gravity 0.91. The window dimensions are shown below and the depth of the centroid is 1.5m below the surface.

\[
\begin{align*}
A_b &= bh \\
I_{G,b} &= \frac{bh^3}{12}
\end{align*}
\]
Mott describes a slightly different approach to the same problem:

- Identify the point where the angle of inclination intersects the free surface (point O)
- Locate the centroid from geometrical arguments (centre of rectangle 0.3 m in)
- Determine $y'$, the depth of the centroid (in this case given as $y' = 1.5$ m)
- Determine $s_1$, the inclined distance to the centroid: $s_1 = \frac{y'}{\sin \theta} = \frac{1.5}{\sin 60^\circ} = 1.73$ m
- Calculate total area of plate: $A = 1.2 \times 0.6 = 0.72$ m$^2$
- Calculate the resultant force

$$F_R = \rho g A y' = \omega A y' = \sigma g \rho_{\text{water}} A y' = 0.91 \times 9.81 \times 10^3 \times 1.5 \times 0.72 = 9.63$ kN$

- Calculate $I_G$, the moment of inertia around centroidal axis
Rectangular window at an angle

At this point, Mott provides an alternative method of derivation of the resultant force and the centre of pressure (rewritten in common notation):

- Force on a small element: \( dF = pdA = wydA = \rho gydA \)
- Work in the plane direction: \( y = s \sin \theta \)
- Force is then \( dF = \rho gs(\sin \theta)dA \)
- Total force found by integration \( F_R = \int_A dF = \int_A \rho gs(\sin \theta)dA = \rho g(\sin \theta)\int_A s dA \)
- From mechanics \( \int_A s dA \) is the product of the area \( A \) and the distance of the centroid from the reference axis (radius of gyration): \( \int_A s dA = s_1 A = k_o A \)
- The resultant force is \( F_R = \rho g(\sin \theta)s_1 A = \rho g(\sin \theta)k_o A \)
- Substituting \( y' = s_1 \sin \theta \) gives \( F_R = \rho g y' A \)
Rectangular window at an angle

Then find the centre of pressure: the position where the resultant force acts so as to have the same effect as the distributed force over the whole surface. Express as a moment around an axis through O an perpendicular to diagram

- Moment of each force element: \( dM = dF \cdot y \)
- Substitute \( dF = \rho gs(\sin \theta) dA \): \( dM = s(\rho gs(\sin \theta) dA) = \rho g \sin \theta (s^2 dA) \)
- The moment of the plane is found by integrating over the whole surface. The resultant force acts on the centre of pressure so that its moment w.r.t. O is \( F_R s_2 \).
- Therefore: \( F_R s_2 = \int_A \rho g \sin \theta (s^2 dA) = \rho g \sin \theta \int_A s^2 dA \)
- From mechanics again, \( \int_A s^2 dA \) is the moment of inertia \( I_o \) of the area about O
- So that \( F_R s_2 = \rho g \sin \theta I_o \) or \( s_2 = \frac{\rho g \sin \theta I_o}{F_R} \)
- Substitute for \( F_R \): \( s_2 = \frac{\rho g \sin \theta I_o}{\rho g \sin \theta s_1 A} = \frac{I_o}{s_1 A} \)
Rectangular window at an angle

- Using the transfer theorem from mechanics (parallel axes): \( I_o = I_G + As_1^2 \) where \( I_G \) is the moment of axis around the centroidal axis:

\[
s_2 = \frac{I_o}{s_1 A} = \frac{I_G + As_1^2}{s_1 A} = \frac{I_G}{s_1 A} + s_1 \quad \text{which can be written} \quad s_2 - s_1 = \frac{I_G}{s_1 A}
\]

- This can be re-written to give everything in terms of depth: \( y' = s_1 \sin \theta \Rightarrow s_1 = \frac{y'}{\sin \theta} \)

\[
D = s_2 \sin \theta = \left[ \frac{I_G}{s_1 A} + s_1 \right] \sin \theta = \left[ \frac{I_G}{y'/\sin \theta A} + \frac{y'}{\sin \theta} \right] \sin \theta
\]

\[
D = \frac{I_G \sin^2 \theta}{Ay'} + y'
\]

since \( I_G = Ak_G^2 \)

\[
D = \sin^2 \theta \frac{I_G}{Ay'} + y' = \sin^2 \theta \frac{k_G^2}{y'} + y'
\]
Rectangular window at an angle

Returning to the problem:

• Calculate \( I_G \), the moment of inertia around centroidal axis

\[
I_{G,\square} = \frac{bh^3}{12} = \frac{1.2 \times 0.6^3}{12} = 0.0216 \text{ m}^4
\]

• Calculate location of centre of pressure

\[
s_2 = \frac{I_G}{s_1 A} + s_1 = \frac{0.0216}{1.73 \times 0.72} + 1.73 = 0.01734 + 1.73 = 1.7473 \text{ m}
\]

watch precision in answer (Mott is wrong).

• Answer is variously:
  17.3mm below centroid
  1.747m below free surface
  at a depth of \( D = s_2 \sin \theta = 1.7473 \sin 60^\circ = 1.513 \text{ m} \)
Piezometric head

In the examples we have considered so far, the upper surface is assumed to be at ambient or atmospheric pressure. Calculations of pressure for the fluid are therefore for gauge pressure \( p_{\text{atmos}} = 0 \). This is appropriate for determining forces on our plates since ambient (atmospheric pressure) acts on the outside of the vessel. The calculations as performed therefore given the **NET FORCE**.

What if the vessel is pressurised and the free surface pressure is not atmospheric pressure?

A convenient method uses the concept of piezometric head in which the actual pressure on the fluid is converted into an equivalent additional depth of fluid.
Piezometric head

The liquid is considered to be deeper by a depth of $h_a$ and open to atmosphere at the top.
Piezometric head

The additional depth is found by converting the pressure (as gauge) to an equivalent head:

\[ p_a = \rho g \Rightarrow h_a = \frac{p_a}{\rho g} = \frac{p_a}{w} \]

This depth is added to any depth in the problem as

\[ y_e = y + h_a \]

specifically giving for the depth of the centroid:

\[ y'' = y' + h_a \]
Rectangular window under pressure

Returning to the previous problem with \( p_a = 10.3 \) kPa:

- Determine \( h_a \):
  \[
  h_a = \frac{p_a}{w} = \frac{10.3 \text{ kN/m}^2}{0.91 \times 9.81 \times 10^3} = \frac{10.3 \text{ kN/m}^2}{8.93 \text{ kN/m}^3} = 1.15 \text{ m}
  \]

- Determine \( y'' \), the depth of the centroid
  \[
  y'' = y' + h_a = 1.5 + 1.15 = 2.65 \text{ m}
  \]

- Calculate the resultant force
  \[
  F_R = \rho g A y'' = 0.91 \times 9.81 \times 10^3 \times 2.65 \times 0.72 = 17.02 \text{ kN}
  \]

- The distance \( s_1 \) changes:
  \[
  s_1' = y''/ \sin \theta = 2.65/ \sin 60^\circ = 3.06 \text{ m}
  \]

- and therefore
  \[
  s_2' = \frac{I_G}{s_1 A} + s_1 = \frac{0.0216}{3.06 \times 0.72} + 3.06 = 3.0698 \text{ m}
  \]

- with the distance between centroid and centre of pressure now 9.8 mm
Pressure diagrams

Resultant force and centre of pressure can be found graphically for vertical walls and other surface of constant vertical height for which it is convenient to calculate the horizontal force per unit width.

Example: vertical wall containing a liquid

The relationship between $p$ and $y$ is linear and the triangle represents the increase in pressure with depth.

The area of this triangle is the product of pressure and depth and will be the resultant force per unit width.
Pressure diagrams

Area of pressure diagram
\[ = \frac{1}{2} BC \times AB = \frac{1}{2} \rho g H \times H \]

Therefore, resultant force is
\[ F_R = \frac{\rho g H^2}{2} \text{ pre unit width} \]

which acts through centroid P which is at a depth of \(2H/3\).

Using the equations:
\[ F_R = \rho g Ay' = \rho g (H \times 1) \times \frac{H}{2} = \frac{\rho g H^2}{2} \]

\[ k_G^2 = \frac{I_G}{A} = \frac{1 \times H^3}{12} \frac{1}{1 \times H} = \frac{H^2}{12} \]

\[ y' = \frac{H}{2} \]

and
\[ D = \sin^2 \theta \frac{k_G^2}{y'} + y' = 1 \times \frac{H^2/12}{H/2} + \frac{H}{2} = \frac{2}{3} H \]

\[ \theta = 90^\circ \Rightarrow \sin^2 \theta = 1 \]
Pressure diagrams can also be drawn for submerged (opening sluice at base of tank) and inclined surfaces (similar to examples shown previously). In both cases, only a partial triangle adds to the area calculation and in the second case, the pressure diagram is drawn along the immersed surface.

\[ p = \rho g H \]
Example

A closed rectangular tank with vertical sides is 1.8m deep and contains water to a depth of 1.2m. Air is pumped into the space above the water until the air pressure is $35\text{kNm}^{-2}$. If the length of one wall of the tank is 3m, determine the resultant force on this wall and the height of the centre of pressure above the base.
Example

Air pressure is applied uniformly across the wall, represented by a rectangular area:

\[ F_{R,\text{air}} = (p_a \times \text{AB}) \times \text{width} = 35 \times 10^3 \times 1.8 \times 3 = 189 \times 10^3 \text{ N} \]

with a centre of pressure 0.9m above the base. Pressure due to the water is applied only up to 1.2m and is represented by the triangular area:

\[ F_{R,\text{air}} = \frac{1}{2} (\rho g H \times \text{DE}) \times \text{width} = \frac{1}{2} \times 10^3 \times 9.81 \times 1.2 \times 1.2 \times 3 = 21.9 \times 10^3 \text{ N} \]

with a centre of pressure at \( H/3 = 0.4 \text{m} \) from base. The total force is then

\[ F_{R,\text{total}} = F_{R,\text{air}} + F_{R,\text{water}} = 189 \times 10^3 \text{ N} + 21.9 \times 10^3 \text{ N} = 210.19 \text{kN} \]

with a centre of pressure found from a moment argument:

\[ F_{R,\text{total}} \times x = F_{R,\text{air}} \times 0.9 \text{m} + F_{R,\text{water}} \times 0.4 \text{m} \]

\[ x = (189 \times 0.9 + 21.9 \times 0.4) / 210.19 = 0.84 \text{m} \]