ELEC3215
FLUIDS AND MECHANICAL MATERIALS

Hydrostatics III – buoyancy
Force on a curved surface

Returning to simple considerations of force but using what we have developed so far for forces on submerged surfaces, we will now consider the case of curved surfaces. If the surface is broken up into small area elements, the forces on these elements due to the pressure are not parallel and must be combined vectorially. It is usually convenient to calculate components separately.

There are two cases: fluid located above the surface and fluid below:
Force on a curved surface

PQ is the immersed surface. Resultant force $F_R$ is separated into horizontal and vertical components ($F_{R,\text{horiz}}$, $F_{R,\text{vert}}$):

(a) Fluid above plane

(b) Fluid below plane
Force on a curved surface

Case (a):

PQ’P₁ is a vertical plane through point P and QQ’ is the horizontal plane bounding the surface of interest. The fluid element PQ’Q is in equilibrium and therefore the resultant force of the element on PQ’ must be equal and opposite to the horizontal component of the force exerted by the fluid on PQ since there are no other forces. In addition, PQ’ is the projection of PQ on the vertical plane PP₁, therefore:

Horizontal component \( F_{R,\text{horiz}} = F_R' \) resultant force on the projection of PQ on a vertical plane

In addition, for equilibrium, \( F_{R,\text{horiz}} \) and \( F_R' \) must act along the same straight line (or rotation would occur). Therefore, the horizontal component \( F_{R,\text{horiz}} \) acts through the centre of pressure of PQ’, the projection of PQ on a vertical plane.
Force on a curved surface

The vertical component $F_{R,\text{vert}}$ is entirely due to the weight of the fluid in the area $PQQ_1P_1$ lying vertically above $PQ$. Since there are no shear forces on $PP_1$ and $QQ_1$ (hydrostatic), there are no other vertical forces.

$$F_{R,\text{vert}} = \text{weight of fluid vertically above } PQ$$

which will act vertically downwards through the centre of gravity $G$ of $PQQ_1P_1$. 
Force on a curved surface

Case (b):

Similarly, P’QQ₁ is a vertical plane through point Q and PP’ is the horizontal plane bounding the surface of interest. The fluid element PP’Q is in equilibrium and therefore the resultant force of the element on P’Q must be equal and opposite to the horizontal component of the force exerted by the fluid on PQ since there are no other forces. In addition, P’Q is the projection of PQ on the vertical plane P’Q₁, therefore:

\[
\text{Horizontal component } F_{R,\text{horiz}} = F'_R \quad \text{resultant force on the projection of PQ on a vertical plane}
\]

In addition, for equilibrium, \( F_{R,\text{horiz}} \) and \( F'_R \) must act along the same straight line (or rotation would occur). Therefore, the horizontal component \( F_{R,\text{horiz}} \) acts through the centre of pressure of P’Q, the projection of PQ on a vertical plane.
Force on a curved surface

If the surface were removed and the volume represented by PQQ₁P₁ were filled with liquid of the same density, then it would be in equilibrium with its own weight and the vertical force on PQ. Therefore

\[ F_{R,\text{vert}} = \text{weight of volume of same fluid which would lie vertically above PQ} \]

which will act vertically upwards through the centre of gravity G of imaginary fluid volume PQQ₁P₁.

Note:

- The points P₁ and Q₁ must lie on the free surface in defining the imaginary volume.
- Again for closed vessels under pressure, an additional head is defined to give an imaginary free surface.
Force on a curved surface

The resultant force is found by combining the components vectorially. In general, the components in three directions may not meet at a point and therefore cannot be represented by a single force. However in 2D as shown in our examples,

The two forces: $F_{R, \text{horiz}}$ and $F_{R, \text{vert}}$ intersect at point O.

Therefore:

$$F_R = \sqrt{F_{R, \text{horiz}}^2 + F_{R, \text{vert}}^2}$$

The resultant force then acts through O at an angle $\theta$ given by:

$$\theta = \tan^{-1} \left( \frac{F_{R, \text{vert}}}{F_{R, \text{horiz}}} \right)$$

In the special case of a cylinder, all forces must be radial and therefore must pass through the centre of curvature O.
Example

Sluice gate comprising a circular arc of radius 6m and angle 60° as shown:

Calculate the magnitude and direction of the resultant force on the gate and the location w.r.t. O of a point on its line of action. The water fills the gate to the top.

Depth of water:

\[ h = 2 \times 6 \sin 30° = 6 \text{ m} \]
Example

Horizontal component of the force on gate is $F_{R,\text{horiz}}$ per unit width (into the page):

\[
\begin{align*}
&= \text{resultant force on line PQ} \\
&= \rho gh \times \frac{h}{2} \\
&= \frac{\rho gh^2}{2} \\
&= \frac{10^3 \times 9.81 \times 6^2}{2} \\
&= 176.58 \text{ kN m}^{-1}
\end{align*}
\]
Example

Vertical component of the force on gate is $F_{R,\text{vert}}$ per unit width (into the page):

$$= \text{weight of water displaced by arc PSQ}
   = \left(\text{arcsection PSQ} - \text{triangle PSQ}\right)_{\text{area}} \rho g
   = \left(\frac{60}{360} \pi 6^2 - 6 \sin 30^\circ \times 6 \cos 30^\circ\right) \times 10^3 \times 9.81
   = \left(6\pi - 9\sqrt{3}\right) \times 10^3 \times 9.81
   = 31.99 \text{ kN m}^{-1}$$
Example

Resultant force on gate:

\[ F_R = \sqrt{F_{R,\text{horiz}}^2 + F_{R,\text{vert}}^2} \]
\[ = \sqrt{176.58^2 + 31.99^2} \]
\[ = 179.46 \text{ kN m}^{-1} \]

acting at an angle of

\[ \theta = \tan^{-1}\left(\frac{F_{R,\text{vert}}}{F_{R,\text{horiz}}}\right) \]
\[ = \tan^{-1}\left(\frac{31.99}{176.58}\right) \]
\[ = \tan^{-1}(0.1816) \]
\[ = 10.27^\circ \]

Note: answer to last part is:

Since surface is cylindrical, then the force must pass through O, the centre of curvature.
Buoyancy

The method of calculating the forces on a curved surface is general. It applies to all shapes of surface and therefore, the surface of a totally submerged object.

For any vertical surface through the object, the projected area of each of the two sides is equal and therefore the horizontal forces are equal and opposite:
Buoyancy

There is therefore no horizontal resultant force on the object due to the surrounding fluid. The only force on the object can therefore be vertical and is referred to as the **buoyancy** or upthrust.

The buoyancy force or upthrust is equal to the difference between the resultant force on the upper and lower surfaces of the object. Assuming that the surface ABCD is horizontal:

\[
\text{upthrust on body} = \text{upward force on lower surface ABCDE} - \text{downward force on upper surface ABCDE'}
\]
Buoyancy

Resolving:

upthrust on body =

= weight of volume of fluid \( A'B'C'D'DABCE \)

− weight of volume of fluid \( A'B'C'D'DABCE' \)

= weight of volume of fluid \( ABCE'DE \)

Giving the general result:

Upthrust on body = weight of fluid displaced by object

This force acts through the centroid of the volume of fluid displaced, known as the CENTRE OF BUOYANCY.

This is known as ARCHIMEDES’ PRINCIPLE
Buoyancy

As an alternative way of looking at this result, if the object were completely replaced by the fluid, the forces acting on the boundaries representing the surface of the object are those which hold the fluid in equilibrium. Therefore, the upward force on the boundary must be equal to the downwards force corresponding to the weight of the fluid contained within that boundary i.e. the weight displaced by the object.
Buoyancy

If we now examine a body immersed so that part of its volume, \( V_1 \), is in a fluid of density \( \rho_1 \) and the remainder of its volume, \( V_2 \), is in a fluid of density \( \rho_2 \).

upthrust on upper volume, \( F_{R_1} = \rho_1 g V_1 \)

acting through \( G_1 \), the centroid of \( V_1 \).

upthrust on lower volume, \( F_{R_2} = \rho_2 g V_2 \)

acting through \( G_2 \), the centroid of \( V_2 \).

Total upthrust \( = (\rho_1 V_1 + \rho_2 V_2) g \)

The positions of \( G_1 \) and \( G_2 \) are not necessarily on the same vertical line and the centre of buoyancy of the whole body is not necessarily the centroid of the whole body.
Example

A rectangular pontoon has a width $b = 6 \text{ m}$, a length $l = 12 \text{ m}$ and a draught (distance from waterline to base) $D = 1.5 \text{ m}$ in fresh water ($\rho = 1000 \text{ kg m}^{-3}$). Calculate:

(a) The weight of the pontoon.

(b) Its draught in seawater ($\rho_{sea} = 1000 \text{ kg m}^{-3}$).

(c) The load which can be supported if maximum draught is 2 m.

When the pontoon is floating unloaded, the upthrust on the immersed volume is equal to the weight of the pontoon. Since the upthrust is also equal to the weight of the displaced fluid:

$$\text{weight of pontoon} = \text{weight of fluid displaced}$$

$$= \rho g b l D$$

(a) in fresh water

$$= 1000 \times 9.81 \times 6 \times 12 \times 1.5$$

$$= 1059.5 \text{ N}$$
Example

(b) the draught in seawater \( (\rho_{\text{sea}} = 1000 \text{ kg m}^{-3}) \) is found by recasting:

\[
draught, \ D = \frac{\text{weight}}{\rho g b l} = \frac{1059.5}{1000 \times 9.81 \times 6 \times 12} = 1.46 \text{ m}
\]

(c) For the maximum draught of 2 m,

\[
\text{total upthrust} = \text{weight of fluid displaced} = \rho g b l D
\]
\[
= 1000 \times 9.81 \times 6 \times 12 \times 2
\]
\[
= 1412.6 \text{ N}
\]

load that can be supported = upthrust – weight of pontoon
\[
= 1412.6 – 1059.5
\]
\[
= 353.1 \text{ kN}
\]
Equilibrium

The equilibrium of floating bodies is maintained if the two forces: the upthrust $F_R$ acting through the centre of buoyancy; and the weight $W = mg$ acting through the centre of gravity, are equal and act along the same straight line. Now, we know that the resultant is equal to the weight of the fluid displaced, which is equal to $\rho g V$, so that:

$$\rho g V = mg \Rightarrow V = \frac{mg}{\rho g} = \frac{m}{\rho}$$

As explained previously, this equilibrium may be stable, unstable or neutral.
Stability

If we have an object totally submerged in a fluid, then again, the upthrust $F_R$ acts through the centre of buoyancy, B, and the weight $W = mg$ acts through the centre of gravity, G. These two points are fixed relative to the object since the shape of the object does not change with orientation.

An angular displacement from equilibrium produces a moment: $W \times BG \times \theta$

There are two cases:
(a) G below B, moment acts to right the object: STABLE
(b) G above B, moment acts to overturn the object: UNSTABLE
Stability of floating objects

If we now return to the floating object:

The upthrust $F_R$ acts through the centre of buoyancy $B$ of the displaced fluid.
The weight $W = mg$ acts through the centre of gravity $G$.
Both act along the same line vertically.

However, when the shape is displaced from the vertical in this case, while the volume of the displaced fluid remains the same, the shape of the displaced fluid and therefore the centre of buoyancy moves, denoted $B_1$ in the figures.
Stability of floating objects

As a result, since upthrust $F_R$ and $W$ no longer act along the same line a turning moment proportional to $W \times \theta$ is produced.

M is the point at which the line of action of $F_R$ cuts the original vertical through the centre of gravity G and is referred to as the METACENTRE. The distance GM is referred to as the metacentric height. The figure above indicates three cases:

1. From (a), if M lies above G, a righting moment $W \times GM \times \theta$ is produced, the body is in stable equilibrium and the metacentric height is considered positive.
2. From (b), if M lies below G, the moment $W \times GM \times \theta$ is overturning, the body is in unstable equilibrium and the metacentric height is considered negative.
3. If M coincides with G, the body is in neutral equilibrium.

For real floating objects such as ships, which can tilt in any direction, the longitudinal (rolling) and transverse (pitching) directions have different metacentres and therefore different metacentric heights.
Position of the metacentre

For a known shape, B and the position of M relative to it can be found:
(from Douglas p78)
Position of the metacentre

For small angles of tilt, \( BM = BB' / \theta \)

The movement of the centre of gravity of displaced fluid (centre of buoyancy) is the result of removal of the wedge \( OAA' \) and the addition of \( OCC' \). Total weight of fluid is still the weight of the vessel:

\[
\text{weight of } OAA' = \text{weight of } OCC'
\]

Volume swept out by \( \delta A \) around \( O \) axis = \( DD' \times \delta A = \delta A x \theta \)

Therefore

\[
\text{weight of } OAA' = \sum_{x=0}^{x=OA} \rho g \delta A x \theta
\]

\[
\text{weight of } OCC' = \sum_{x=0}^{x=OC} \rho g \delta A x \theta
\]

and

\[
\rho g \theta \sum_{x=0}^{x=OA} \delta A x = \rho g \theta \sum_{x=0}^{x=OC} \delta A x \Rightarrow \sum \delta A x = 0
\]
Position of the metacentre

Since \( \sum \delta A_x \) is the first moment of waterline plane around O axis, the axis must pass through the centroid of the plane.

The distance BB’ can now be calculated since the couple produced by the movement of the weight of OAA’ to OCC’ must be equal to that produced by moving \( F_R \) from B to B’.

\[
I = \sum \delta A_x^2
\]

\[
\text{moment due to altered displacement} = \sum_{x=OA}^{x=OC} \rho g \delta A x \theta \times x = \rho g \theta \sum \delta A_x^2 = \rho g \theta I
\]

Moment due to movement of \( F_R \)

\[
= F_R \times BB' = \rho g V \times BB'
\]

Therefore \( \rho g V \times BB' = \rho g \theta I \) and \( \text{BM} = \frac{BB'}{\theta} = \frac{I}{V} \)

\( BB' = \frac{\theta I}{V} \)

The metacentric radius
A cylindrical buoy, weighing 10 kN floating in seawater, is loaded with 2 kN. What is the maximum height of the centre of gravity of the load for stability?
The volume of liquid displaced is $V$ and total buoyancy force $= \rho g V = \rho g \pi \left(\frac{d}{2}\right)^2 Z$

In equilibrium, this force must equal the total weight: $W + W_1 = \rho g \pi \left(\frac{d}{2}\right)^2 Z$

Depth of immersion: $Z = 4 \frac{W + W_1}{\rho g \pi d^2} = 4 \frac{10 + 2}{1025 \times 9.81 \times \pi \times 1.8^2} \times 10^3 = 0.47 \text{ m}$

Centre of buoyancy = centre of gravity of displaced volume: $OB = Z/2 = 0.235 \text{ m}$

For the limiting case described, the buoy and the load will *just* be in stable equilibrium, i.e. the metacentre $M$ is at the centre of gravity of the combined object $G'$. 
Metacentre height $G'M = 0$ and $BG' = BM$:

$$BM = \frac{I}{V} = \frac{\frac{\pi}{4} \left(\frac{d}{2}\right)^4}{\frac{\pi}{4} d^2 Z} = \frac{\pi}{64} \frac{d^4}{d^2 Z} = \frac{d^2}{16Z} = \frac{1.8^2}{16 \times 0.47} = 0.431\text{m}$$

The position of $G'$ is $Z' = \frac{Z}{2} + BG' = 0.235 + 0.431 = 0.666\text{m}$

Obtain value of $Z_1$ for this value of $Z'$ using moments: $W_1Z_1 + ZW = (W_1 + W)Z'$

Maximum height: $Z_1 = \frac{(W_1 + W)Z' - ZW}{W_1} = \frac{W_1Z' + W(Z' - Z)}{W_1} = Z' + \frac{W}{W_1}(Z' - Z)$

$$= 0.666 + \frac{10 \times 10^3}{2 \times 10^3}(0.666 - 0.45) = 0.666 + 5 \times 0.216 = 1.746\text{m}$$