ELEC3215
FLUIDS AND MECHANICAL MATERIALS

Fluid mechanics II – Momentum
The continuity equation or the conservation of mass equation is one of the fundamental equations underpinning fluid mechanics. The second is the momentum equation, which relates the applied forces to the acceleration of the fluid or the rate of change of momentum. This fundamental equation of motion is how the motion of the fluid is related to what we do to it.

The momentum of an object is the product of mass and velocity. The fluid elements which make up a flow stream will have momentum and whenever there is a change in velocity, there must be a corresponding change in momentum.

From Newton’s laws, this is produced by a force which may come from a wall, or a machine in the system and the fluid exerts an equal and opposite force on the corresponding object. Since these forces are to do with the movement of the fluid, they are referred to as dynamic and act in addition to the static forces.
The momentum equation

The momentum equation is determined by considering the flow of fluid through the entrance and exit of the pipe. Fluid flow is steady and non-uniform.

Newton’s second law states that any change in velocity, implying a change in momentum must require a force. In the case of the expanding pipe, to determine the change in momentum, continuity says that for our control volume (the pipe), the mass flow rate is

\[ \dot{m} = \rho Q = \rho_1 A_1 u_1 = \rho_2 A_2 u_2 \]
The momentum equation

In other words, there is no storage in pipe. The rate at which momentum passes through a x-section is

$$\rho A u \times u$$

with a value of $\rho_1 A_1 u_1 \times u_1$ for the entrance and $\rho_2 A_2 u_2 \times u_2$ for the exit. The rate of change of momentum through the pipe is

$$\rho_2 A_2 u_2 u_2 - \rho_1 A_1 u_1 u_1$$

which can be written (using continuity to substitute) as

$$\rho_1 A_1 u_1 u_2 - \rho_1 A_1 u_1 u_1 = \rho_1 A_1 u_1 (u_2 - u_1) = \dot{m}(u_2 - u_1)$$

or mass flow per unit time multiplied by the change in velocity. This is the INCREASE in momentum in the direction of motion and there must be a force on the fluid such that

$$F = \dot{m}(u_2 - u_1)$$

By Newton’s third law, the fluid must exert this force back on its surroundings.
The momentum equation

The momentum equation for two and three-dimensional flow along a streamline is for the general case, where the fluid velocity entering and leaving our control volume are not aligned along the same axis. We will illustrate this in two dimensions, with
The momentum equation

Both momentum and force are vector quantities and act at different angles to the \( x \)-axis as shown. The resultant force is found by applying the momentum equation to each component direction separately.

\[
F_x = \text{rate of change of momentum in } x\text{-direction} \\
= \text{mass flow rate} \times \text{change of velocity in } x\text{-direction} \\
= \dot{m}(u_1 \cos \theta_1 - u_2 \cos \theta_2) \\
= \dot{m}(u_{1,x} - u_{2,x})
\]

Similarly

\[
F_y = \dot{m}(u_1 \sin \theta_1 - u_2 \sin \theta_2) \\
= \dot{m}(u_{1,y} - u_{2,y})
\]

These components are combined to give the resultant force:

\[
F = \sqrt{F_x^2 + F_y^2} \\
\angle F = \tan^{-1}\left(\frac{F_y}{F_x}\right)
\]
The momentum equation

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= \dot{m}(u_{1,x} - u_{2,x})
\]

Similarly \( F_y = \dot{m}(u_1 \sin \theta_1 - u_2 \sin \theta_2) \)

\[= \dot{m}(u_{1,y} - u_{2,y})\]

These components are combined to give the resultant force:

\[
F = \sqrt{F_x^2 + F_y^2} \\
\angle F = \tan^{-1}\left(\frac{F_y}{F_x}\right)
\]
The momentum equation

For the third dimension, the same procedure can be used (although the directional calculation is different) so that

\[ F_z = \dot{m}(u_{1,z} - u_{2,z}) \]

and \[ |\mathbf{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2} \], \[ \hat{\mathbf{F}} = \left( \frac{F_x}{\sqrt{F_x^2 + F_y^2 + F_z^2}}, \frac{F_y}{\sqrt{F_x^2 + F_y^2 + F_z^2}}, \frac{F_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \right) \]

In summary:

- Total force exerted on the rate of change of momentum in the fluid in a control volume = of the fluid passing through the control volume in a given direction

  \[ \mathbf{F} = \dot{m}(\mathbf{u}_{out} - \mathbf{u}_{in}) \]

The positive direction of \( \mathbf{F} \) is the same as the positive direction of \( \mathbf{u} \).
The momentum equation

For any control volume, the total force is made up of different component force contributions, which individually may appear in different components of the resultant force and should therefore be considered separately:

The fluid in the control volume may experience three different types of forces:

- That from contact with a solid body or boundary, $F_1$
- The action of a body force which acts on the fluid e.g. gravity, $F_2$
- Internal action of the fluid outside the control volume, $F_3$

The total momentum equation is then

$$\mathbf{F} = F_1 + F_2 + F_3 = m(u_{out} - u_{in})$$

The force $F_R$ exerted on solid objects or boundaries will be equal and opposite only to the first component force i.e.

$$F_R = -F_1$$
The momentum correction factor

The equations we have derived so far assumed an ideal fluid as previously discussed. In a real fluid, there is a velocity profile and therefore the fluid velocity cannot be assumed to be uniform. The momentum change per unit time for the whole flow is found by dividing the cross-sectional area into sufficiently small sections that the flow in the section can be assumed to be uniform and then summing over all elements of the cross-section. So that

mass passing through element $\delta A$ per unit time $= \rho \delta A u$

momentum per unit time passing through element $= \text{mass per unit time} \times \text{velocity}$

$= \rho \delta A u \times u$

$= \rho \delta A u^2$

total momentum per unit time $= \int_A \rho u^2 dA$

A knowledge of the flow profile is required to evaluate this integral.
The momentum correction factor

We will examine solutions to flow profiles later but as an example here, we can examine turbulent flow (closest to ideal) in a pipe.

In a pipe of radius $R$, the flow profile is given by Prandtl’s one seventh power law:

$$u = u_{\text{max}} \left( \frac{r'}{R} \right)^{\frac{1}{7}}$$

where $r' = R - r$ is the distance from the wall.
The momentum correction factor

We use the same annular analysis as before, using a ring of radius \( r \) and width \( \delta r \) as the cross-sectional area element in which the velocity is constant:

\[
\delta A = 2\pi r \delta r
\]

For the whole pipe:

\[
\text{total momentum per unit time} = \int_A \rho u^2 dA
\]

\[
= \int_0^R \rho u_{\text{max}}^2 \left( \frac{r'}{R} \right)^{\frac{2}{7}} 2\pi r dr
\]

\[
= \left( \frac{2\pi \rho u_{\text{max}}^2}{R^\frac{2}{7}} \right) \int_0^R (r')^{\frac{2}{7}} r dr
\]

We can substitute \( r = R - r' \) and \( dr = -dr' \) adjust the limits.
The momentum correction factor

The integral becomes:

\[
\text{total momentum per unit time} = \frac{2\pi \rho u_{\text{max}}^2}{R^{2/7}} \int_0^R (r')^{2/7} (R - r')( - dr')
\]

\[
= \frac{2\pi \rho u_{\text{max}}^2}{R^{2/7}} \int_0^R \left((r')^{9/7} - R(r')^{2/7}\right) dr'
\]

\[
= \frac{2\pi \rho u_{\text{max}}^2}{R^{2/7}} \left[ \frac{7}{16} (r')^{16/7} - \frac{7R}{9} (r')^{9/7} \right]_0^R
\]

\[
= \frac{2\pi \rho u_{\text{max}}^2}{R^{2/7}} R^{16/7} \left( \frac{7}{9} - \frac{7}{16} \right)
\]

\[
= \frac{49\pi}{72} \rho R^2 u_{\text{max}}^2
\]
The momentum correction factor

As previously discussed, it is normal to use mean velocity rather than maximum velocity:

\[
\bar{u} = \frac{\text{total volume per unit time passing through cross-section } Q}{\text{total area of cross-section } A} = \frac{1}{\pi R^2} \int_0^R u \delta A
\]

substitute \( \delta A = 2\pi r \delta r \) and \( u = u_{\max} \left( \frac{r'}{R} \right)^\frac{1}{7} \)

\[
\bar{u} = \frac{1}{\pi R^2} \int_0^R u_{\max} \left( \frac{r'}{R} \right)^\frac{1}{7} 2\pi r dr = \frac{2u_{\max}}{R^{15/7}} \int_0^R (r')^\frac{1}{7} rdr
\]

again substitute \( r = R - r' \) and \( dr = -dr' \) adjust the limits.
The momentum correction factor

The average velocity is then

\[
\bar{u} = \frac{2u_{\text{max}}}{R^{15/7}} \int_{0}^{R} (r')^{1/7} (R - r')(dr')
\]

\[
= \frac{2u_{\text{max}}}{R^{15/7}} \int_{0}^{R} \left( (r')^{8/7} - R(r')^{1/7} \right) dr'
\]

\[
= \frac{2u_{\text{max}}}{R^{15/7}} \left[ \frac{7}{15} (r')^{15/7} - \frac{7R}{8} (r')^{8/7} \right]_{0}^{R}
\]

\[
= \frac{2u_{\text{max}}}{R^{15/7}} R^{15/7} \left( \frac{7}{8} - \frac{7}{15} \right)
\]

\[
= \frac{49}{60} u_{\text{max}}
\]

giving the reciprocal relationship

\[
u_{\text{max}} = \frac{60}{49} \bar{u}
\]
The momentum correction factor

Substituting:

\[
\text{total momentum per unit time} = \frac{49\pi}{72} \rho R^2 \left( \frac{60}{49} \bar{u} \right)^2
\]

\[
= \frac{49}{72} \left( \frac{60}{49} \right)^2 \rho \pi R^2 \bar{u}^2
\]

\[
= \frac{50}{49} \rho \pi R^2 \bar{u}^2
\]

\[
= 1.02 \rho \pi R^2 \bar{u}^2
\]

We can compare this with the mass per unit time

\[
\text{mass per unit time} = \rho \times \text{area} \times \text{average velocity}
\]

\[
= \rho \pi R^2 \bar{u}
\]
The momentum correction factor

If we had assumed an ideal fluid, we would have obtained the expression:

\[
\text{total momentum per unit time} = \rho \pi R^2 \bar{u}^2
\]

In our example, given the mass per unit time \( \rho \pi R^2 \bar{u} \)

\[
\text{total momentum per unit time} = 1.02 \times \text{mass per unit time} \times \text{mean velocity}
\]

The number here is the \textit{momentum correction factor}, usually denoted \( \beta \) so that

\[
\text{total momentum per unit time} = \beta \times \text{mass per unit time} \times \text{mean velocity}
\]
Example 1: force from a jet

If we have a jet striking a plane surface, then we can examine the force on the plate by considering a control volume around the plate and stationary relative to it. The problem is then simply that of a jet striking a stationary plate.

This is the same as saying that the boundary condition on the fluid velocity is

$$u_{normal} = (u - v)\cos\theta$$
Example 1: force from a jet

The mass flow rate into the control volume is also relative:

\[ \dot{m} = \rho A(u - v) \]

\[ \rightarrow \dot{m} = \rho Au \quad \text{if stationary} \]

The rate of change of momentum normal to surface:

\[ \frac{d}{dt} (\text{momentum}) = \rho A(u - v) \times (u - v) \cos \theta = \rho A(u - v)^2 \cos \theta \]

\[ \rightarrow \rho Au^2 \cos \theta \quad \text{if stationary} \]

\[ \rightarrow \rho Au^2 \quad \text{if perpendicular} \]

This requires a force to generate the change in momentum and there must be an equal and opposite reaction force on the plate:

\[ F_{\text{normal}} = \rho A(u - v)^2 \cos \theta \]
Consider a jet of water of velocity 10ms\(^{-1}\) from a nozzle of diameter 5cm, striking a stationary flat plate perpendicularly. Assuming that the surface is frictionless, calculate the change in force if (a) the angle of the plate is changed to 45\(^o\) and (b) if the plate is moved at a speed of 2ms\(^{-1}\) towards the nozzle.

Initially, the force is
\[ F_{\text{normal}} = \rho A u^2 \cos \theta = 10^3 \pi (0.025)^2 10^2 1 = 196.35N \]

(a) The angle is now 45\(^o\) : 
\[ F_{\text{normal}} = \rho A u^2 \cos \theta = 10^3 \pi (0.025)^2 10^2 \cos 45 = 138.84N \]

so the force magnitude is reduced by 57.5N and the direction has changed.

(b) The velocity of the plate is -2ms\(^{-1}\) :
\[ F_{\text{normal}} = \rho A(u - v)^2 \cos \theta = 10^3 \pi (0.025)^2 (10 - (-2))^2 1 \\ = 10^3 \pi (0.025)^2 12^2 1 = 282.74N \]

an increase in force of 86.4N.
Example 2: force due to a curved plate

With velocity and momentum both vector quantities, changes in direction without accompanying change in magnitude also produces resultant forces. As an example, consider the curved vane deflecting a fluid stream without impact shock.

Assuming the mass flow rate of the stream is 0.8kgs\(^{-1}\) and that the fluid has a mean velocity of 30ms\(^{-1}\) before encountering the vane and 25ms\(^{-1}\) afterwards, calculate the force on the vane. Ignore the effects of gravity.
Example 2: force due to a curved plate

The general momentum equation is

\[ \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \dot{m}(u_{out} - u_{in}) \]

In our case, we neglect gravity and assume that the pressure in a jet is uniform everywhere

\[ F_{R,x} = -F_1 = \dot{m}(u_{in,x} - u_{out,x}) \]
\[ F_{R,y} = -F_1 = \dot{m}(u_{in,y} - u_{out,y}) \]

The nozzle and vane are fixed relative to each other, mass leaving the control volume must be equal to mass entering (mass conservation). Substituting for the velocities:

\[ F_{R,x} = -F_1 = \dot{m}(u_1 - u_2 \cos \theta) \]
\[ F_{R,y} = -F_1 = \dot{m}(0 - u_2 \sin \theta) = -\dot{m}(u_2 \sin \theta) \]
Example 2: force due to a curved plate

Substituting values:

\[ F_{R,x} = \dot{m}(u_1 - u_2 \cos \theta) \]
\[ = 0.8(30 - 25 \cos 60^\circ) \]
\[ = 14 \text{N} \]

\[ F_{R,y} = -\dot{m}(u_2 \sin \theta) \]
\[ = -0.8(25 \sin 60^\circ) \]
\[ = 17.32 \text{N} \]

The resultant force is therefore:

\[ F_R = \sqrt{F_{R,x}^2 + F_{R,y}^2} = \sqrt{14^2 + 17.32^2} = 22.27 \text{N} \]

at an angle

\[ \theta_1 = \tan^{-1} \left( \frac{F_{R,y}}{F_{R,x}} \right) = \tan^{-1} \left( \frac{17.32}{14} \right) = 51.05^\circ \]
Example 3: force on pipe bends

A related problem is that of the force on the bend of a pipeline or other similar aspect of a system. There are static force components from the pressures and dynamic forces from the momentum change. Calculate the total force.

Pipe inlet diameter 500mm, tapering to outlet of 250mm. Angle $\theta$ is 45°. Input pressure is 40kNm$^{-2}$ and outlet pressure is 23kNm$^{-2}$. Fluid is oil of density 850kgm$^{-3}$ and flow rate of 0.45m$^3$s$^{-1}$.
Example 3: force on pipe bends

Align the axes $x$ and $y$ as shown to simplify. The control volume is the pipe:

mass flow entering = $\rho Q$

The force on the fluid is

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \dot{m}(u_{out} - u_{in})$$

where $F_1$ is the force exerted on the fluid by the walls, $F_2$ we will assume to be zero and $F_3$ due to pressures $p_1$ and $p_2$ at the inlet and outlet. The force exerted on the pipe will be $F_R = -F_1$. In the $x$-direction

$$F_{1,x} + F_{3,x} = \dot{m}(u_{2,x} - u_{1,x}) \Rightarrow F_{R,x} = -F_{1,x} = F_{3,x} - \dot{m}(u_{2,x} - u_{1,x})$$

$$= p_1 A_1 - p_2 A_2 \cos \theta - \rho Q(u_2 \cos \theta - u_1)$$

and in the $y$-direction

$$F_{1,y} + F_{3,y} = \dot{m}(u_{2,y} - u_{1,y}) \Rightarrow F_{R,y} = -F_{1,y} = F_{3,y} - \dot{m}(u_{2,y} - u_{1,y})$$

$$= p_2 A_2 \sin \theta + \rho Q u_2 \sin \theta$$
Example 3: force on pipe bends

For our problem

\[ A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} 0.5^2 = 0.196m^3, \quad A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} 0.25^2 = 0.0491m^3 \]

\[ Q = 0.45m^3s^{-1} \Rightarrow \left\{ \begin{array}{l} u_1 = \frac{Q}{A_1} = \frac{0.45}{0.196} = 2.292ms^{-1} \\ u_2 = \frac{Q}{A_2} = \frac{0.45}{0.0491} = 9.167ms^{-1} \end{array} \right. \]

So that

\[ F_{R_x} = p_1 A_1 - p_2 A_2 \cos \theta - \rho Q(u_2 \cos \theta - u_1) \]

\[ = 40 \times 10^3 \times 0.196 - 23 \times 10^3 \times 0.0491 \cos 45^\circ \]

\[ - 850 \times 0.45(9.167 \cos 45^\circ - 2.292) \]

\[ = 10^3(7.855 - 0.798 - 1.603) \]

\[ = 5.454kN \]
Example 3: force on pipe bends

and

\[ F_{R,y} = p_2 A_2 \sin \theta + \rho Qu_2 \sin \theta \]
\[ = 23 \times 10^3 \times 0.0491 \sin 45^\circ + 850 \times 0.45 \times 9.167 \sin 45^\circ \]
\[ = 10^3 (0.798 + 2.479) \]
\[ = 3.277 \text{kN} \]

The resultant force is therefore:

\[ F_R = \sqrt{F_{R,x}^2 + F_{R,y}^2} = \sqrt{5.454^2 + 3.277^2} = 6.362 \text{N} \]

at an angle

\[ \theta_1 = \tan^{-1}\left( \frac{F_{R,y}}{F_{R,x}} \right) = \tan^{-1}\left( \frac{3.277}{5.454} \right) = 31^\circ \]
Example 4: jet reaction force

Another example is the simple jet reaction force, where the momentum of a fluid is increased from one section to another. Newton’s third law says that for this to occur, there must be a force acting back on the system producing the change in momentum.

If we consider the simple example of a jet of water of diameter 60mm emerging from a hole in the side of a water filled tank, 2m below the surface of the water in the tank, which has an average velocity of 5.5 ms\(^{-1}\). Determine the force on the tank and its contents when it is (a) stationary and (b) moving with a velocity of 1.2 ms\(^{-1}\) in the direction opposite to the jet, with the velocity of the jet relative to the tank unchanged. In the latter case, determine the rate of work.

The tank is kept filled to the same height and we will assume that gravity and pressure gradient effects can be ignored.
Example 4: jet reaction force

Assuming that gravity and pressure gradient effects can be ignored, then the only force acting in the direction of motion is the jet reaction force. So that the force on the system is

\[ F = -\dot{m}(u_{\text{out}} - u_{\text{in}}) \]

reaction force from jet = mass discharge rate \times increase in velocity in direction of jet. Mass discharged per unit time =

\[ \rho A u = \rho \pi \left( \frac{d}{2} \right)^2 u_{\text{jet}} = 1000\pi \times \left( \frac{0.06}{2} \right)^2 5.5 = 15.55 \text{ kgs}^{-1} \]

(a) stationary tank, \( u_{\text{out}} = 5.5 \text{ m/s} \) and \( u_{\text{in}} = 0 \) in the direction opposite to the jet.

\[ \therefore F = -15.55(5.5 - 0) = 85.5 \text{ N} \]
Example 4: jet reaction force

(b) moving tank, \( u_{\text{out}} = (1.2 + 5.5) \text{m/s} \) and \( u_{\text{in}} = 1.2 \) in the direction opposite to the jet.

\[
F = -15.55(6.7 - 1.2) = -15.55(5.5) = 85.5N
\]

And work rate is work done per second or force times velocity

\[
85.5 \times 1.2 = 102.6W
\]
Drag force on a flat plate

Solutions of fluid problems are obtained using a combination of the different governing equations.

Example, when a fluid flows over a stationary flat surface, there is a shear stress between the surface and the fluid which retards the fluid. This is the viscous drag effect which results in the boundary layer as discussed previously. The drag force acting on the plate from the fluid can be found using the momentum equation.

At a distance in advance of the plate, the fluid moves at a constant velocity of $U$. After the fluid impacts on the leading edge of the plate at position O, a boundary layer will form which increases in thickness as distance along the plate increases. This boundary layer is caused by the shear stress to between the plate and the fluid and results in a variation in velocity over a thickness of $\delta$ from the plate in the lateral direction. This thickness $\delta$ is a function of distance $x$ from the leading edge of the plate.

The drag force $F_D$, also increases with distance as the region of affected fluid increases. The thickness of the boundary layer $d$ is typically defined as being the height at which $u = 0.99U$. 
Drag force on a flat plate

The force can be determined by examining the control volume PQRS shown in the figure, which consists of a section of the boundary layer of length $\Delta x$ at a distance of $x$ from the leading edge $O$. 

![Diagram of a flat plate with notation and forces](image-url)
Drag force on a flat plate

The fluid enters the control volume through PQ and QS and leaves through RS. The momentum equation is:

\[
\text{Force acting on the fluid in the control volume in } x\text{-direction} = \text{Rate of increase of momentum in } x\text{-direction of fluid passing through control volume}
\]

For a given depth \(D\) into the diagram, we can determine the momentum flux through the vertical faces of the volume by integrating the momentum flux in the small element \(\delta y\) over the height of the boundary layer.

\[
\text{momentum per second passing through element at RS} = \text{mass per second } \times \text{velocity} = \rho D \delta y u_2 \times u_2
\]

Therefore, total momentum flux through RS is

\[
\rho D \int_0^{\delta_2} u_2^2 \, dy
\]
Drag force on a flat plate

The momentum flux through PQ is similarly \( \rho D \int_0^{\delta_1} u_1^2 dy \)

Continuity of flow for surface QS gives:

rate of fluid flow into the control volume through QS = rate of flow through RS - rate of flow through PQ

i.e. \( Q_{QS} = D \int_0^{\delta_2} u_2 dy - D \int_0^{\delta_1} u_1 dy \)

with \( x \)-component of momentum entering through QS \( = \rho Q_{QS} U = \rho D \left( \int_0^{\delta_2} u_2 dy - \int_0^{\delta_1} u_1 dy \right) U \)

Giving for the total momentum in the \( x \)-direction:

\[
\rho D \left( \int_0^{\delta_2} u_2^2 dy - \int_0^{\delta_1} u_1^2 dy - U \left( \int_0^{\delta_2} u_2 dy - \int_0^{\delta_1} u_1 dy \right) \right)
\]
Drag force on a flat plate

Equating the $x$-component of the force exerted on the fluid by the boundary:

$$-\tau_o D \Delta x = \rho D \left( \int_0^{\delta_2} u_2^2 \, dy - \int_0^{\delta_1} u_1^2 \, dy - U \left( \int_0^{\delta_2} u_2 \, dy - \int_0^{\delta_1} u_1 \, dy \right) \right)$$

$$= \rho D \left( \int_0^{\delta_2} (u_2^2 - U u_2) \, dy - \int_0^{\delta_1} (u_1^2 - U u_1) \, dy \right)$$

variation in $(u^2 - U u)$ across the control volume

$$\Delta \left[ \int_0^{\delta} (u^2 - U u) \, dy \right]$$
Drag force on a flat plate

Therefore:

\[-\tau_o D\Delta x = \rho D\Delta \left[ \int_0^\delta u(u-U)dy \right] \]

In the limit \( \Delta x \to 0 \):

\[\tau_o = \rho U^2 \frac{d}{dx} \int_0^\delta \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy \]

The force on the plate is then found from

\[F_D = D \int_0^x \tau_o \, dx \]

doubled if the plate is considered to be two sided.
Euler’s equation of motion

Another equation of motion which is commonly used is due to Euler. Consider the short streamtube section shown in the figure, with sufficiently small cross-section that the velocity is uniform across it.
Euler’s equation of motion

Mass flow per unit time: \[ = \rho Au \]

Rate of increase of momentum from AB to CD \[ = \rho Au[(u + \delta u) - u] = \rho Au\delta u \]

Forces acting:
- Force from \( p \) in direction of motion \[ = pA \]
- Force from \( p + \delta p \) opposing motion \[ = (\rho + \delta \rho)(A + \delta A) \]
- Force from \( p_{side} \) in direction of motion \[ = p_{side}A \]
- Force from \( mg \) opposing motion \[ = mg \cos \theta \]

Total force \[ = pA - (p + \delta p)(A + \delta A) + p_{side}A - mg \cos \theta \]
Euler’s equation of motion

The side pressure is a linear function from \( p \) to \( p + \delta p \):

\[ p_{\text{side}} = p + k\delta p \]

The weight of the element

\[ mg = \rho g \times \text{volume} = \rho g \left( A + \frac{\delta A}{2} \right) \delta s \]

and \( \cos \theta = \frac{\delta z}{\delta s} \)

Resultant force in direction of motion is

\[ F = -A\delta p - \rho g A \delta z \]
Euler’s equation of motion

Applying Newton’s second law gives:

$$\rho Au \delta u = -A \delta p - \rho g A \delta z$$

Divide by $\rho A \delta s$ \\Rightarrow \quad \frac{1}{\rho} \frac{\partial p}{\partial s} + u \frac{\partial u}{\partial s} + g \frac{\partial z}{\partial s} = 0$

or as $\delta s \to 0$ :

$$\frac{1}{\rho} \frac{dp}{ds} + u \frac{du}{ds} + g \frac{dz}{ds} = 0$$

This is Euler’s equation in differential form, giving the relationship between pressure, velocity, density and elevation along a streamline. It cannot be integrated until the relationship between density and pressure is known.
Bernoulli’s equation

For an incompressible fluid, the density is constant and integrating Euler’s equation along the streamline w.r.t. $s$ gives

$$\frac{p}{\rho} + \frac{u^2}{2} + g\frac{z}{g} = \text{constant}$$

which has terms which represent energy per unit mass. Dividing by $g$ gives

$$\frac{p}{\rho g} + \frac{u^2}{2g} + \frac{z}{g} = \text{constant}$$

which is known as Bernoulli’s equation giving the relationship (per unit weight) between pressure velocity and elevation for steady flow of a frictionless fluid of constant density with alternate form (useful in thermodynamics) of

$$p + \frac{1}{2} \rho u^2 + z \rho g = \text{constant}$$

which is more obviously energy per unit volume.
Bernoulli’s equation represents a crude form of conservation of energy (which we will return to) so that

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

for any two positions along the streamline.

For a compressible fluid, partial integration gives:

$$\int \frac{dp}{\rho g} + \frac{u^2}{2g} + z = \text{constant}$$

which requires the relationship between density and pressure to be known. We will discuss this further in the later sections on thermodynamics.
The Navier-Stokes’ equations

The Navier-Stokes’ equations are the differential forms of the continuity equation and momentum equations. They are typically widely used as the basis for the solution of fluid mechanics problems. From before, the continuity equation is

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z} = 0
\]

The differential form of the momentum equation is derived as follows.

We begin by examining the same cuboidal element as before and consider the momentum change and forces (stresses) acting on it.
In the $x$-direction, the forces are:

\[
\begin{align*}
\rho u_x & + \frac{\partial}{\partial x}(\rho u_x) \delta x \\
\sigma_x & + \frac{\partial}{\partial x} \sigma_x \delta x \\
\tau_{xz} & + \frac{\partial}{\partial z} \tau_{xz} \delta z \\
\tau_{xy} & + \frac{\partial}{\partial y} \tau_{xy} \delta y \\
\end{align*}
\]
The Navier-Stokes' equations

The acceleration in the $x$-direction is

$$\frac{\delta u_x}{\delta t} = u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} + \frac{\partial u_x}{\partial t}$$

The rate of change of momentum is then:

$$\rho \delta x \delta y \delta z \left( u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} + \frac{\partial u_x}{\partial t} \right)$$

The force acting on the element is then the sum of the normal stress on the facing surfaces and the lateral shear stresses acting on the side faces.
The Navier-Stokes’ equations

The net force in the x-direction is then:

\[ F_x = \rho f_x \delta x \delta y \delta z + \left[ \sigma_x - \left( \sigma_x + \frac{\partial}{\partial x} \sigma_x \delta x \right) \right] \delta y \delta z \]

\[ - \left[ \tau_{xz} - \left( \tau_{xz} + \frac{\partial}{\partial z} \tau_{xz} \delta z \right) \right] \delta x \delta y - \left[ \tau_{xy} + \left( \tau_{xy} - \frac{\partial}{\partial y} \tau_{xy} \delta y \right) \right] \delta x \delta z \]

\[ = \left( \rho f_x - \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) \delta x \delta y \delta z \]

where \( f_x \) denotes the external body force acting on the element (which could be gravitational, magnetic, electrical and so on). The element is assumed to be sufficiently small that changes in stress or mass flow are linear.
The Navier-Stokes’ equations

The momentum equation is (in its complete form) the vector equation incorporating the three dimensions, found by equating this force and the rate of change of momentum:

\[
\left( \rho f_x - \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) \delta x \delta y \delta z = \rho \delta x \delta y \delta z \left( u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} + \frac{\partial u_x}{\partial t} \right)
\]

\[
\left( \rho f_y - \frac{\partial \sigma_y}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) = \rho \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right)
\]

and similarly for the vector components in the other dimensions:

\[
\left( \rho f_y + \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) = \rho \left( \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} \right)
\]

\[
\left( \rho f_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} - \frac{\partial \sigma_z}{\partial z} \right) = \rho \left( \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \right)
\]
The Navier-Stokes’ equations

• These equations are general but cannot be integrated without representations of the stresses acting on the fluid element.

• For inviscid flow, there is no shear stress and the normal stress can simply be replaced by the pressure and these equations reduce to Euler’s equation if the body force is simply gravity.

To further examine the behaviour of the fluid, we examine the properties of Newtonian fluids. The normal and shear stresses are related to the velocity gradients by the coefficients of viscosity:

• We have already discussed the dynamic viscosity $\mu$ as the constant of proportionality between velocity gradient and the viscous stress

• We introduce here the second viscosity coefficient $\lambda$ which is the constant of proportionality between the stress and the volumetric deformation (the sum of the velocity gradients in each of the three dimensions)
The Navier-Stokes’ equations

The stress velocity gradient expressions or constitutive equations are:

\[
\begin{align*}
\sigma_x &= p - 2\mu \frac{\partial u_x}{\partial x} - \lambda \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right), & \tau_{xy} &= \mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \\
\sigma_y &= p - 2\mu \frac{\partial u_y}{\partial y} - \lambda \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right), & \tau_{xz} &= \mu \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \\
\sigma_z &= p - 2\mu \frac{\partial u_z}{\partial z} - \lambda \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right), & \tau_{yz} &= \mu \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right)
\end{align*}
\]

The effect of the second viscosity coefficient is small in practice, but a good approximation is given by the **Stokes’ Hypothesis**:

\[
\lambda = -\frac{2}{3} \mu
\]

The pressure then is the average of the normal stress in the equations above.
The Navier-Stokes' equations

Returning to the momentum equation (assuming homogeneous fluids):

\[
\left( \rho f_x - \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) = \rho \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right)
\]

We substitute the stress expressions (combined with Stokes' hypothesis)

\[
\sigma_x = p - 2\mu \frac{\partial u_x}{\partial x} - \frac{2}{3} \mu \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right), \quad \tau_{xy} = \mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)
\]

and write the right hand side of the equation as the substantive derivative:

\[
\rho \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right) = \rho \frac{Du_x}{Dt} = \rho \frac{du_x}{dt} + (\mathbf{u} \cdot \nabla)u_x
\]
The Navier-Stokes’ equations

The left hand side of the equation is by substitution

\[
\left( \rho f_x - \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right)
\]

\[
= \rho f_x - \frac{\partial p}{\partial x} + 2 \mu \frac{\partial^2 u_x}{\partial x^2} - \frac{2}{3} \mu \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + \mu \left[ \frac{\partial}{\partial y} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \right]
\]

\[
= \rho f_x - \frac{\partial p}{\partial x} + 2 \mu \frac{\partial^2 u_x}{\partial x^2} - \frac{2}{3} \mu \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + \mu \left[ \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} + \frac{\partial}{\partial y} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \right]
\]

\[
= \rho f_x - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_x}{\partial x^2} + \mu \left[ \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right] - \frac{2}{3} \mu \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + \mu \frac{\partial}{\partial x} \frac{\partial u_x}{\partial x} + \mu \left[ \frac{\partial}{\partial x} \frac{\partial u_y}{\partial y} + \frac{\partial}{\partial x} \frac{\partial u_z}{\partial z} \right]
\]

\[
= \rho f_x - \frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right] - \frac{2}{3} \mu \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + \mu \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)
\]

\[
= \rho f_x - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_x}{\partial x^2} + \mu \frac{\partial^2 u_x}{\partial y^2} + \mu \frac{\partial^2 u_x}{\partial z^2}
\]
The Navier-Stokes’ equations

Equating right and left gives the full compressible momentum equations:

\[
\rho \frac{Du_x}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) + \frac{1}{3} \mu \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)
\]

\[
\rho \frac{Du_y}{Dt} = \rho f_y - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \right) + \frac{1}{3} \mu \frac{\partial}{\partial y} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)
\]

\[
\rho \frac{Du_z}{Dt} = \rho f_z - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) + \frac{1}{3} \mu \frac{\partial}{\partial z} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)
\]

or in vector notation:

\[
\rho \frac{Du}{Dt} = \rho \frac{du}{dt} + \rho (u \cdot \nabla)u = \rho f - \nabla p + \mu \nabla^2 u + \frac{1}{3} \mu \nabla (\nabla \cdot u)
\]
The Navier-Stokes’ equations

If the fluid is incompressible, then using continuity:
\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \nabla \cdot \mathbf{u} = 0
\]

\[
\rho \frac{Du_x}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right)
\]

\[
\rho \frac{Du_y}{Dt} = \rho f_y - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \right)
\]

\[
\rho \frac{Du_z}{Dt} = \rho f_z - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right)
\]

\[
\rho \frac{D\mathbf{u}}{Dt} = \rho \frac{d\mathbf{u}}{dt} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \rho \mathbf{f} - \nabla p + \mu \nabla^2 \mathbf{u}
\]

These are the Navier-Stokes’ equations and are widely used to model fluid flow behaviour.
The Reynold’s number is the ratio of the inertial forces and the viscous forces, which in the Navier Stokes’ equation are

\[ \rho (u \cdot \nabla) u \quad \text{and} \quad \mu \nabla^2 u \]

Scaling:

\[ \frac{\rho (u_o)^2}{l_o} \quad \text{and} \quad \frac{\mu u_o}{(l_o)^2} \]

The ratio is:

\[ \frac{\rho (u \cdot \nabla) u}{\mu \nabla^2 u} \sim \frac{\rho (u_o)^2 (l_o)^2}{\mu u_o l_o} \]

Giving as before:

\[ \text{Re} = \frac{\rho u_o l_o}{\mu} \]