Steady two-dimensional fluid flow through a chamber of height $2d$. The chamber has a width ($z$-direction) much greater than the height $2d$, so that the system can be considered two-dimensional. The flow is plane and parallel, of the form $u = (u(x,y,t), 0, 0)$ which automatically satisfies the continuity equation.

Laminar flow example

Navier Stokes:

\[
\frac{\partial \rho u}{\partial t} = -\nabla p + \frac{\partial}{\partial y} \left( \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right)
\]

\[
\frac{\partial \rho v}{\partial t} = -\nabla p + \frac{\partial}{\partial y} \left( \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right)
\]

\[
\frac{\partial \rho}{\partial t} = -\nabla p + \frac{\partial}{\partial y} \left( \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right)
\]

\[
\frac{\partial \rho}{\partial t} = -\nabla p + \frac{\partial}{\partial y} \left( \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right)
\]

Integrating twice with respect to $y$, we get

\[
u = -\frac{\rho}{2\mu} y^2 + C_1 y + C_2
\]

Applying the boundary conditions, $u = 0$ at $y = \pm d$ gives

\[C_1 = 0, \quad C_2 = \frac{\rho d^2}{2\mu}
\]
Laminar flow example

The final solution is

\[ u_y = \frac{R}{2\mu L} (y^2 - y^2) \]

Solution is the equation of a parabola:

with maximum velocity \( u_{max} = \frac{R}{2\mu L} \)

average velocity \( u_{avg} = \frac{R}{3\mu L} \)

and volume flow rate per unit width given by:

\[ Q = \int_{-d}^{d} u_y dy = \frac{2R}{3\mu L} \]

Time dependent – boundary layers

Behaviour of an impulsively moved boundary

\[ \frac{\partial y}{\partial t} = \frac{\partial y}{\partial x} = 0 \]

No applied pressure and \( p \) does not depend on \( x \)

\[ \frac{\partial p}{\partial x} = 0 \]

Therefore

\[ \frac{\partial u_y}{\partial x} = \frac{\mu}{\rho} \frac{\partial u_y}{\partial y} = \frac{\partial u_y}{\partial y} \]

with initial conditions \( u_y(x, 0) = 0, \ y > 0 \)

and boundary conditions \( u_y(0, t) = U, \ t > 0 \)

Boundary layers

Seek similarity solution

\[ y \rightarrow ay, \ t \rightarrow at', \ \Rightarrow u = f(\zeta), \ \zeta = \frac{y}{(at)^{1/2}} \]

Giving

\[ t' = \frac{1}{2a} \]

Integrating once

\[ t' = C_1 e^{-\zeta} \]

The initial and boundary conditions become

\[ f(0) = 0, \ f(0) = U \]

Giving the solution:

\[ u_y = U \left[ 1 + \frac{1}{2\pi} \int_0^\infty e^{-\zeta} d\zeta \right] \]

\[ \zeta = \frac{y}{(at)^{1/2}} \]
Boundary layers

Estimated thickness

Steady pipe flow

For any pipe or duct with symmetry axis parallel to the \(x\)-axis

\(u\) is parallel to the \(x\)-axis and is only a function of \(y\) and \(z\):

\[
\mathbf{u} = (u(y, z), 0, 0)
\]

\[
\frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} = 0 \implies \text{pressure is constant across the pipe}
\]

Navier-Stokes equations (\(x\)-component)

Laplace's equation with constant pressure drop per unit length:

\[
\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\rho} \frac{\partial p}{\partial x}
\]

Circular pipe

Converting the Laplacian into circular polar coordinates:

Integrating gives

Since the velocity is finite in the centre of the channel,

Applying the boundary condition, \(u = 0\) at \(r = R\)

giving the general solution:

which is, again, the equation for a parabola.
Circular pipe - momentum

Mass flow rate: a mass equal to \( \rho 2 \pi r_u \, dr \) flows through the annular element

\[
\dot{m} = 2 \pi \rho \int r_u \, dr
\]

giving

\[
\dot{m} = \frac{\Delta P \pi r^4}{8 \mu}
\]

This type of flow is referred to as Hagen-Poiseuille flow or simple Poiseuille flow.

The \( R^4 \) proportionality indicates the huge increase in pressure required to maintain a flow rate as the radius of the pipe decreases.

Alternate derivation - Newton

The viscous drag force between the surfaces of the hollow cylinders:

\[
F = \mu \frac{d u_r(r)}{d r}
\]

Inner surface area = \( 2 \pi r_i \)

Outer surface area = \( 2 \pi (r + \Delta r) \)

Pressure drop per metre = \( \frac{P}{l} \)

Assuming \( \Delta r \ll r \), area of end of hollow cylinder = \( 2 \pi r \Delta r \)

Net force on hollow cylinder (unit length) from applied pressure = \( -\frac{P}{l} 2 \pi r \Delta r \)

Alternate derivation

The applied pressure force must be exactly balanced (in the steady state) by the difference between the viscous drag force on the outer and inner surface of the hollow cylinder.

\[
\frac{P}{l} 2 \pi r \Delta r = 2 \pi (r + \Delta r) \frac{d u_r(r + \Delta r)}{d r} - 2 \pi r \frac{d u_r(r)}{d r}
\]

\[
\frac{P}{l} = \frac{u_r(r + \Delta r) - u_r(r)}{\Delta r}
\]

in the limit \( \Delta r \to 0 \)

Integrating gives as before

\( u_r(r) = \frac{P}{4 \mu} r^2 + C_1 \ln r + C_2 \)
Flow in a real pipe

Using the momentum equation for an angled pipe section:

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Flow in a real pipe

For the annular element shown, the momentum equation gives \( p \) is pressure and \( \tau \) is shear stress

\[
p_2\pi r_2 dr - (p + \tau_2 \pi r_2) dr = \left(\frac{d}{dx} \pi r_2^2 \right) dx + W \sin \theta - W g z
\]

where \( W = \text{weight} \). Set \( W = 2\pi r_2 dx \sin \theta \) and \( \sin \theta = -\frac{dz}{dx} \).

\( z \) is elevation. Equation becomes

\[
p_2\pi r_2 dr - p_2\pi r_2 dr - \frac{d}{dr} \left( 2\pi r_2 dr \right) dr - 2\pi r_2 dr W g z dx = 0
\]

or

\[
\frac{d}{dx} \left( p + \tau_2 \pi r_2 \right) + \frac{1}{r} \frac{d}{dr} \left( \tau_2 r_2 \right) = 0
\]

or

\[
\frac{d}{dx} \left( p + pgz \right) + \frac{1}{r} \frac{d}{dr} \left( \tau - K \right) = 0
\]

Flow in a real pipe

The first term is independent of \( r \) and this can be integrated simply to give:

\[
\frac{1}{2} \frac{d}{dx} \left( p + pgz \right) + \tau + K = 0
\]

Where \( K \) is a constant. When \( \tau \) is zero in this expression, \( K = 0 \).

Then, using Newton’s law of viscosity with the converse relationship between \( \tau \) and distance measure from wall \( y \):

\[
\tau = \frac{\rho u}{2} \left( \rho u + pgz \right) + \frac{\rho u u}{2} = \frac{\rho u}{2}
\]

giving

\[
\frac{1}{2} \frac{d}{dx} \left( p + pgz \right) = \frac{\rho u}{2} \frac{d}{dx} - K
\]

\[
dr = \left[ \frac{r}{2\pi} \frac{d}{dx} \left( p + pgz \right) + \frac{K}{\rho u} \right] dr
\]
Flow in a real pipe

Integrate with respect to $r$:

$$u = \frac{1}{4\mu} \frac{d}{dt} (p + \rho \gamma) + \frac{K}{\mu} \ln r + K_0$$

Since velocity is finite in centre, again $K = 0$ and the velocity is zero for $r = R$ we get

$$K = -\frac{R^2}{4\mu} \frac{d}{dt} (p + \rho \gamma)$$

So that

$$u = \frac{r^2 - R^2}{4\mu} \frac{d}{dt} (p + \rho \gamma)$$

Flow in a real pipe

The maximum flow rate is at $r = 0$:

$$u_{max} = \frac{R^2}{4\mu} \frac{d}{dt} (p + \rho \gamma)$$

The volume flow rate is calculated by integrating the incremental flow through the annulus

$$\delta Q = u 2\pi r \delta r$$

From $r = 0$ to $R$ so that

$$Q = \int 2\pi r \delta r = \frac{\pi}{2\mu} (p + \rho \gamma) \left( R^2 - r^2 \right) r \delta r$$

$$= \frac{\pi}{2\mu} (p + \rho \gamma) \frac{R^3}{3} - \left( \frac{R^3}{3} - \frac{r^3}{3} \right)$$

$$= \frac{\pi}{2\mu} (p + \rho \gamma) R^3$$

Flow in a real pipe

The average flow rate is $Q/A$

$$\bar{u} = \frac{\pi}{8\mu} (p + \rho \gamma) R^4 / \pi R^4 = \frac{R^4}{8\mu} (p + \rho \gamma) - \frac{1}{2} u_{max}$$

The flow rate can also be written in terms of a pressure drop $\Delta p$ over length $l$ in a pipe of diameter $d$:

$$Q = \frac{d \Delta p d^3}{128 \mu l}$$

And rearranged to give the Hagen-Poiseuille equation:

$$\Delta p = \frac{128 \mu Q}{d^2} \bar{u} = 32 \mu \bar{u} = \frac{\pi}{4}$$
Duct flow summary

Quick summary

Duct or pipe flow classification:

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Turbulent</th>
<th>Laminar</th>
</tr>
</thead>
</table>

- Ideal: Increasing Reynolds number, more "ideal".
- Turbulent: Increasing Reynolds number, increasing friction, more loss.
- Laminar: Increasing Reynolds number, increasing pressure.

Turbulent flow in a pipe (Prandtl):

\[ \nu = \nu_{\max} \left( \frac{1}{T} \right) \]

Laminar flow in a pipe (Hagen-Poiseuille):

\[ Q = \frac{\rho \pi d^4}{128 \mu L} \]

Head loss in a closed section pipe (Darcy-Weisbach equation):

\[ h_f = \frac{4L \pi^2}{34} \frac{\nu^2}{d^4} \]

This relationship is often summarised by a Moody Diagram.