To use the complex form of the Fourier series to obtain amplitude, phase and power spectrums for periodic signals. These are in the form of graphical plots and are line spectra. For non-periodic signals the Fourier transform, if applicable, is used instead of the Fourier series.
Background

- Key Question: How is the energy/power of a signal distributed over time/frequency?
- In this section we answer this question for deterministic periodic signals using the Fourier series.
- The answer is most usefully presented in the frequency domain and involves line spectra.
In this section we examine methods of analysis due to the French physicist Jean Baptiste Fourier (1768-1830). In these methods, a system input signal is represented as **sums of continuous sinusoids and superposition** is used to determine the system output to this input signal. Two cases arise here.

- If the signals are **periodic**, the sum is discrete with the sinusoids existing at frequencies which are integer multiples of the repetition, or fundamental, frequency and are known as harmonics. We can describe such signals as having **discrete spectra**.

- If the signals are **aperiodic** the sum is replaced by an integral over a continuous frequency variable and this class of signals all have **continuous spectra**.
According to Fourier’s theorem, any periodic signal waveform defined by the relationship

\[ x(t + T) = x(t) \]

can be represented by its **Fourier series**, the so-called **trigonometric version** of which is

\[ x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \quad (1) \]
Fourier Series

The coefficients $a_n, b_n$ are given by

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos(n\omega_0 t) \, dt$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin(n\omega_0 t) \, dt$$

where $\omega_0 = 2\pi f_0$ and $f_0 = \frac{1}{T}$ is the repetition or fundamental frequency, $n$ is an integer, and hence $nf_0$ denotes the $n$th harmonic frequency. The real constants $a_n$ and $b_n$ are known as the Fourier coefficients of $x(t)$ and $a_0$ is the mean value of the signal $x(t)$.
The Fourier series for the square wave

\[ x(t) = \begin{cases} 
1.0, & t \in \left[-\frac{T}{2}, \frac{T}{2}\right] \\
-1.0, & t \notin \left[-\frac{T}{2}, \frac{T}{2}\right] 
\end{cases} \]
—Fourier Series

is

\[ x(t) = \frac{4}{\pi} \left[ \sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \cdots \right] \]
Fourier Series

Partial Sums.

\[ s_0 = \frac{4}{\pi} \sin \omega_0 t \]
\[ s_2 = \frac{4}{\pi} \left[ \sin \omega_0 t + \frac{1}{3} \sin 3 \omega_0 t \right] \]
\[ s_3 = \frac{4}{\pi} \left[ \sin \omega_0 t + \frac{1}{3} \sin 3 \omega_0 t + \frac{1}{5} \sin 5 \omega_0 t \right] \]
Fourier Series

- The basic elements (sines and cosines) in this Fourier representation are **orthogonal** - see below - and this property is used to prove (1).

- Sufficient conditions for the convergence of the right-hand side of (1) exist and are known as the **Dirichlet** conditions. These state that if $x(t)$ is periodic and piecewise continuous in the interval $-\frac{T}{2} < t < \frac{T}{2}$ and has a left and right hand derivative at each point in this interval then its Fourier series converges and its sum is $x(t)$ if the function is continuous at $t$. If the function is not continuous at $t$ then the sum is the average of the left and right-hand limits of $x(t)$ at $t$. This is supported by the graphs of the partial sums in the last figure.
Fourier Series

- When a function is approximated by a partial sum of a Fourier series there is considerable error in the vicinity of a discontinuity no matter how many terms are included. This is known as the Gibbs phenomenon and is explained as follows. Considering the example again, the continuous terms of the series try to simulate the sudden jump or discontinuity and as the number of terms in the partial sum is increased the ripples are ‘squashed’ towards the discontinuity but the ‘overshoot’ does not reduce to zero.
The Fourier series (1) may be written as

\[ x(t) = a_0 + \sum_{n=1}^{\infty} M_n \cos(\omega_0 nt - \psi) \]  

(2)

where

\[ M_n = \sqrt{a_n^2 + b_n^2} \]

\[ \psi = \tan^{-1}\left(\frac{b_n}{a_n}\right) \]

where \( M_n \) is termed the \textbf{amplitude} of frequency \( nf_0 \) and \( \psi \) is termed the \textbf{phase (lag)} of frequency \( nf_0 \).
Alternative expressions to both (1) and (2) can be developed using the identity

\[ e^{\pm j\theta} = \cos \theta \pm j \sin \theta \]

Applying this result to (1) yields the so-called complex exponential form as

\[
x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j n \omega_0 t}
\]  

(3)
Fourier Series

The right-hand side of (3) must be real since the left-hand side is a real valued function and if we let

\[ c_0 = a_0 \]
\[ c_n = \frac{a_n - j b_n}{2} \]

Then

\[ c_n^* = c_{-n} = \frac{a_n + j b_n}{2} \]

and hence

\[ x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j n \omega_0 t} \] (4)

where

\[ c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn \omega_0 t} \, dt, \ n \neq 0 \] (5)
Fourier Series

Equation (5) may be derived directly from (4) by multiplying this equation by $e^{-jm\omega_0 t}$, integrating over one period, and using the orthogonality property

$$\int_0^T e^{j(n-m)\omega_0 t} \, dt = \begin{cases} 0, \ n \neq m \\ T, \ n = m \end{cases}$$
Using the complex form of the Fourier series, we can now introduce the spectrum of a periodic function $x(t)$. A plot of magnitude $|c_n|$ versus frequency is termed the amplitude spectrum of $x(t)$. A plot of argument $\arg c_n = \angle c_n$ versus frequency is termed the phase spectrum. These are discrete line spectra (where in the figure below it has been assumed that $c_0 \geq 0$).
Fourier Series

Amplitude Spectrum

\[ |c_n| \quad \text{(even)} \]

\[ \text{arg } c_n \quad \text{(odd)} \]

\[ c_0 \geq 0 \]
Fourier Series

The mean normalized power $P$ for a periodic signal is given by

$$P = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t) \, d\,t$$

or, on writing $x^2(t) = x(t)x^*(t)$ (where $*$ denotes the complex conjugate transpose),

$$P = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x^*(t) \, d\,t$$

$$= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{m} \sum_{n} c_n e^{in\omega_0 t} c_m^* e^{-jm\omega_0 t} \, d\,t$$

$$= \frac{1}{T} \sum_{n} \sum_{m} \int_{-\frac{T}{2}}^{\frac{T}{2}} c_n c_m^* e^{i(n-m)\omega_0 t} \, d\,t = \sum_{n=\infty}^{+\infty} |c_n|^2$$

where we have again used orthogonality.
This last result is a version of Parseval’s theorem and has the physical interpretation that the average (or mean normalized) power of the periodic signal $x(t)$ may be regarded as the sum of the power associated with individual frequency components. A plot of $|c_n|^2$ versus frequency is termed the power spectrum and is a decomposition of the power of the function over frequency.
Fourier Series

Power Spectrum

\[ |c_n|^2 \]

\( f = \frac{-5}{T} \frac{-4}{T} \frac{-3}{T} \frac{-2}{T} \frac{-1}{T} \frac{0}{T} \frac{1}{T} \frac{2}{T} \frac{3}{T} \frac{4}{T} \frac{5}{T} \)
Fourier Series

It is possible to simplify the calculation of the Fourier coefficients by exploiting any symmetry properties of the periodic waveform. In particular,

A function is even if \( x(t) = x(-t) \).

A function is odd if \( x(t) = -x(-t) \).

It now follows that if \( x(t) \) is even then in its Fourier series representation \( b_n = 0 \) \( \Rightarrow \) Fourier series has only cosine terms.

If \( x(t) \) is odd then \( a_n = 0 \) \( \Rightarrow \) Fourier series only has sine terms.
Fourier Series

In the case of an even function

\[ a_n = \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \cos(n\omega_0 t) \, dt \]

and in the case of an odd function

\[ b_n = \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \sin(n\omega_0 t) \, dt \]

Note: The odd and even properties of a signal are not intrinsic, i.e. they depend on the relation of the signal to the axes of the co-ordinate system.
Fourier Series

Even AA’, Odd BB’, Neither CC’

AA’ - EVEN
BB’ - ODD
CC’ - NEITHER

3 CHOICES OF VERTICAL AXIS
Fourier Series

If a periodic signal is applied to a linear system with transfer function $G(s)$, and hence frequency response $G(j\omega)$, then each of the harmonic components of this signal will be modified in magnitude and phase by the values of $|G(j\omega)|$ and $\angle G(j\omega)$. In particular, if the input signal is given by

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

then the output signal is also periodic with, in general, the same period (provided $G(s)$ does not have a zero at $s = j\omega = j2\pi f_0$) which can be expressed as

$$y(t) = \sum_{n=-\infty}^{+\infty} G(j2\pi nf_0) c_n e^{j2\pi nf_0 t}$$
Fourier Series — Example

A signal of period $T$ is defined by

$$x(t) = \begin{cases} 0, & -\frac{T}{2} \leq t < 0 \\ H, & 0 < t \leq \frac{T}{2} \end{cases}$$

where $H$ is a positive real number. Obtain the complex form of the Fourier series for this signal and hence sketch its amplitude and power spectrums.
### Fourier Series — Example

**Complex Fourier series coefficients**

\[ c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega_0 t} \, dt, \quad n \neq 0 \]

and for \( n = 0 \)

\[ c_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \, dt \]

Hence in the case given here

\[ c_0 = \frac{H}{2} \]

and for \( n \neq 0 \)

\[ c_n = \begin{cases} \frac{H}{jn\pi}, & \text{n odd} \\ 0, & \text{n even} \end{cases} \]
Amplitude spectrum — plot of $|c_n|$ versus $f$. — Figure 1.

![Figure 1](attachment:image.png)
Power spectrum — plot of $|c_n|^2$ versus $f$. — Figure 2.