This paper contains 4 questions

Answer three questions.

An outline marking scheme is shown in brackets to the right of each question.

This examination contributes 100% of the marks for the module.

University approved calculators MAY be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

14 page examination paper.
Question 1.

(a) A signal with period $T$ is described by

$$x(t) = \begin{cases} 
0, & 0 \leq t < t_0 - \frac{\beta}{2} \\
H, & t_0 - \frac{\beta}{2} \leq t \leq t_0 + \frac{\beta}{2} \\
0, & t_0 + \frac{\beta}{2} < t \leq T 
\end{cases}$$

Obtain the complex form of the Fourier series for this signal.

You may use the following formula where all symbols have their usual meanings

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega_0 t} \, dt, \ n \neq 0.$$  

[8 marks]

(b) Sketch the amplitude and power spectrums for the signal of the previous part of this question.

[8 marks]

(c) Determine the Fourier transform of the rectangular pulse

$$x(t) = \begin{cases} 
\sqrt{L}, & -\frac{L}{2} \leq t \leq \frac{L}{2} \\
0, & \text{elsewhere}
\end{cases}$$

Consider the triangular pulse

$$x_1(t) = \begin{cases} 
1 - \frac{|t|}{L}, & -L \leq t \leq L \\
0, & \text{elsewhere}
\end{cases}$$

Determine the Fourier transform of this pulse either directly or by making use of a property of the Fourier transform.

[8 marks]

(d) A continuous-time system with input $u$ and output $y$ is governed by the differential equation.

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) = b_1 \dot{u}(t) + b_0 u(t)$$
Compute the energy spectral density of the output when the input is the signal $x_1(t)$ of the previous part of this question. What is the energy spectral density if the output $y(t)$ of this system is used as the input to another system with output $z(t)$ governed by the differential equation

$$\ddot{z}(t) + c_1 \dot{z}(t) + c_2 z(t) = d_0 y(t)$$

[6 marks]
Indicative Solution for Question 1.

Answers to Question 2

(a) [8 marks]

\[ c_0 = \frac{1}{T} \int_{t_0}^{t_0 + \frac{T}{2}} H dt = \frac{H \beta}{T} \]

\[ c_n = \frac{1}{T} \int_{t_0}^{t_0 + \frac{T}{2}} H e^{-jn\omega_0 t} dt, \quad n \neq 0 \]

\[ = \frac{H}{n\pi} e^{-jn\omega_0 t_0} \sin \left( \frac{n\omega_0}{2} \right) \]

(b) [8 marks]

Amplitude spectrum — plot of \(|c_n|\) against frequency — line spectra.
Power spectrum — plot of \(|c_n|^2\) against frequency — line spectra.

(c) [8 marks] For \(x(t)\) direct evaluation of the defining integral gives

\[ X(\omega) = \sqrt{L_2} \frac{\sin \left( \frac{n\omega_0}{2} \right)}{\omega_0} \]

By evaluation of the defining integral and since it is an even function

\[ X_1(\omega) = 2 \int_0^L (1 - \frac{t}{L}) \cos (\omega t) dt = \sqrt{L} \left[ \frac{\sin \left( \frac{n\omega_0}{2} \right)}{\omega_0} \right]^2 \]

\(x_1(t)\) can be obtained by the convolution of \(x(t)\) with itself. Hence its Fourier transform is \(X(\omega) \times X(\omega)\)

(d) [6 marks]

In the first case the transfer-function is

\[ G(s) = \frac{s + b_0}{s^2 + a_1 s + a_2} \]

Fourier transform of \(u(t) = x_1(t)\) from the previous part of the question.

In Fourier transform terms

\[ Y(f) = G(f)U(f) \]

Energy spectral density is \(|Y(f)|^2\).

In the second case the transfer-function is

\[ H(s) = \frac{d_0}{s^2 + c_1 s + c_2} \]
In Fourier transform terms

\[ Z(f) = H(f)Y(f) = H(f)G(f)U(f) \]

Energy spectral density is \(|Z(f)|^2\).
Question 2.

(a) Explain what role is played by the following formulas in analogue filter design, where all symbols have their usual meanings:

\[ |G(j\omega)|^2 = \frac{1}{1 + b_n\omega^{2n}}, \quad \Theta_k = (2k - 1)\pi/2n, \quad k = 1, \ldots, 2n. \]

\[ T_{n+1}(\omega) = 2\omega T_n(\omega) - T_{n-1}(\omega) \]

with

\[ T_0(\omega) = 1, \quad T_1(\omega) = \omega \]

[6 marks]

(b) A particular application requires a low-pass Butterworth filter with cut-off frequency of 1 kHz. Determine the transfer-functions for both a 3rd and 4th order filter. Express each transfer-function in polynomial form and for the 4th order case also with a denominator polynomial that is the product of two quadratic polynomials. Compare the attenuation of the two filters in dB at a frequency equal to twice the cut-off frequency.

[12 marks]

(c) After evaluation of the designs in the previous part of this question, it is decided to replace the 3rd order Butterworth filter with a low-pass filter of the same order that gives an equi-ripple in the pass-band of \( \delta \) dB. Construct the transfer function of such a filter. Explain also how the following formula, where all symbols have their usual meanings, can be used to construct another alternative. Your answer must include a clearly labeled sketch explaining how all variables in this formula arise

\[ n \geq \frac{\log[(10^{0.1\alpha_s} - 1)/(10^{0.1\alpha_p} - 1)]}{2\log(\omega_0/\omega_s)} \]

[8 marks]
(d) Explain how to convert a low-pass filter with unity cut-off frequency to a band-stop filter with stop-band between 600 rad/sec and 950 rad/sec. You may use the following formulas, where all symbols have their usual meanings:

\[ s \rightarrow \beta \left( \frac{s}{\omega_0} + \frac{\omega_0}{s} \right), \quad \beta = \frac{\omega_0}{\omega_2 - \omega_1}; \quad \omega_0 = \sqrt{\omega_1 \omega_2} \]

\[ s \rightarrow \frac{1}{\beta} \left( \frac{s}{\omega_0} + \frac{\omega_0}{s} \right)^{-1} = \frac{\omega_0 s}{\beta (s^2 + \omega_0^2)} \]

[4 marks]
Indicative Solution for Question 2.

Answers to Question 3

(a) [6 marks] \(|G(j\omega)|^2\) — magnitude of the Butterworth filter (where we also take \(b_n = 1\)).

\(\Theta_k\) — determines the poles of the Butterworth filter (corresponding to those choices of \(\Theta_k\) which give stable poles.

The recursive formulas for \(T_n(\omega)\) are used to construct the Chebyshev polynomial and hence the filter with

\[ |G(j\omega)|^2 = G(s)G(-s)_{s=j\omega} = \frac{1}{1 + \epsilon^2 T_n^2(\omega)} \]

(b) [12 marks] For the 3rd order Butterworth filter, the stable poles correspond to \(\Theta_1 = \frac{\pi}{6} = \frac{\pi}{2}, \Theta_3 = \frac{3\pi}{6} = \frac{\pi}{2}\). The filter transfer-function scaled to \(\omega_0 = 2000\pi\) rad/sec is

\[ G(s) = \frac{\omega_0^3}{s^3 + 3\omega_0^2 s^2 + 3\omega_0^3 s + \omega_0^5} \]

For the 4th order Butterworth filter, the stable poles correspond to \(\Theta_1 = \frac{\pi}{8}, \Theta_2 = \frac{3\pi}{8}, \Theta_3 = \frac{5\pi}{8}\) and \(\Theta_4 = \frac{7\pi}{8}\).

In factored quadratic form with scaling to the same value of \(\omega_0\) is

\[ G(s) = \frac{\omega_0^3}{(s^2 + 0.7654\omega_0 s + \omega_0^2)(s^2 + 1.847\omega_0 s + \omega_0^2)} \]

In polynomial form

\[ G(s) = \frac{1558.54 \times 10^{12}}{s^4 + 16.42 \times 10^3 s^3 + 134.69 \times 10^6 s^2 + 648.2 \times 10^8 s + 1558.54 \times 10^{12}} \]

Attenuation, use

\[ |G(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^{2n}} \]

For \(\frac{\omega}{\omega_0} = 2\) this gives an attenuation of 18.12 dB for the 3rd order filter and 24.1 dB for the 4th order filter.

(c) [8 marks]

3rd order low-pass Chebyshev filter is required with

\[ |G(j\omega)|^2 = G(s)G(-s)_{s=j\omega} = \frac{1}{1 + \epsilon^2 T_n^2(\omega)} \]

Use the recursive formal of the first part of this question to obtain \(T_3(\omega)\), i.e.,

\[ T_3(\omega) = 4\omega^3 - 3\omega \]

Second part, discussion around the figure below

(d) [4 marks] \(\omega_1 = 600, \omega_2 = 950\) and use the second of the supplied formulas.
Question 3.

(a) A discrete random signal is given by

\[ x[n] = A \cos(\omega_0 n + \psi) + w[n] \]

where \( A \) is the deterministic amplitude, \( \omega_0 \) is the deterministic frequency, \( \psi \) is the randomly distributed phase and \( w[n] \) is zero mean Gaussian white noise with unity variance. Show that the autocorrelation sequence \( r_{xx}(k) \) is

\[ r_{xx}(k) = \frac{1}{2} A^2 \cos(\omega_0 k) + \delta(k) \]

where \( \delta(k) \) is the Kronecker delta function. Compute the power spectral density of this signal.

[10 marks]

(b) Consider again the signal of the previous part of this question. Construct the \( 4 \times 4 \) autocorrelation matrix \( R \) and verify that it can be written in the form

\[
R = \frac{1}{4} A^2 \begin{bmatrix}
1 & e^{j\omega_0} & e^{-j\omega_0} \\
e^{j2\omega_0} & e^{-j2\omega_0} & 1 \\
e^{j3\omega_0} & e^{-j3\omega_0} & e^{j\omega_0}
\end{bmatrix}
\begin{bmatrix}
1 & -e^{-j\omega_0} & e^{-j2\omega_0} & e^{-j3\omega_0} \\
e^{j\omega_0} & e^{j2\omega_0} & e^{j3\omega_0} & 1 \\
1 & e^{j2\omega_0} & e^{j3\omega_0} & e^{j\omega_0}
\end{bmatrix} + I_4
\]

where \( I_4 \) denotes the \( 4 \times 4 \) identity matrix.

[14 marks]

(c) Detail the structure of the eigenvalues and eigenvectors of the matrix \( R \) of the previous part of this question. What information concerning the underlying signal does this structure provide? Also detail how the frequency \( \omega_0 \) can be estimated from \( R \).

[6 marks]
Indicative Solution for Question 3.

Answers to Question 4

(a) [10 marks] First write

\[ x[n] = \frac{1}{2} A(e^{j(\omega_0 k + \psi)} + e^{j(\omega_0 k + \psi)}) + w[n] \]

Then

\[ r_{xx}(k) = E(x[n]x^*[n - k]) + \frac{1}{4} A^2(e^{j\omega_0 k} + e^{-j\omega_0 k}) + \delta(k) = \frac{1}{2} A^2 \cos(\omega_0 k) + \delta(k) \]

\[ P_{xx}(\omega) = \sum_{k=-\infty}^{\infty} r_{xx}(k)e^{-j\omega k} \]

\[ = \frac{1}{2} \pi A^2(u_0(\omega - \omega_0) + u_0(-\omega - \omega_0)) + 1 \]

where \( u_0(\omega) \) is a delta spike in the \( \omega \) plane of period \( 2\pi \).

Let \( \mu_\omega \neq 0 \) denote the mean of \( x[n] \). Then \( r_{xx}(k) \) is increased by \( \mu(k)^2 \) for each \( k \).

(b) [14 marks]

\[ R = \frac{1}{2} A^2 \left[ \begin{array}{cccc} 1 & \cos(\omega) & \cos(2\omega) & \cos(3\omega) \\ \cos(\omega) & 1 & \cos(2\omega) & \cos(3\omega) \\ \cos(2\omega) & \cos(\omega) & 1 & \cos(2\omega) \\ \cos(3\omega) & \cos(2\omega) & \cos(\omega) & 1 \end{array} \right] + I_4 \]

The required factorization now follows by inspection.

(c) [6 marks]

\( R \) has rank equal to two and has two eigenvalues larger than one and three equal to the noise power of unity. This gives the number of signals and the noise power. Let \( u_1 \) and \( u_2 \) be the eigenvectors corresponding to the two largest eigenvalues. Then these vectors span the signal subspace and the other three, \( u_3, u_4 \) and \( u_5 \), the noise subspace. Estimates for \( \omega_0 \) and \( -\omega_0 \) can be obtained from either subspace.
Question 4.

(a) Draw a clearly labelled diagram of a generic adaptive filter and explain what each part actually does. Also give clearly labelled adaptive filter architectures for the following:
   i) noise cancellation;
   ii) system identification;
   iii) inverse system identification/equalisation. [12 marks]

(b) The Wiener-Hopf equation is

\[ w_{opt} = R^{-1}p, \]

where all symbols have their usual meanings. State the mean squared error problem solved by this equation and give formulas for the matrix \( R \) and vector \( p \). [8 marks]

(c) Let \( x[n] \) and \( d[n] \) be zero-mean real-valued random processes with the following auto and cross-correlations.

\[
\begin{align*}
  r_{dd}(k) &= \begin{cases}
    1, & k = 0 \\
    0, & \text{elsewhere}
  \end{cases}, &
  r_{xx}(k) &= \begin{cases}
    2, & k = 0 \\
    1, & |k| \geq 1
  \end{cases} \\
  r_{dx}(k) &= \begin{cases}
    1, & k = 0 \\
    0, & \text{elsewhere}
  \end{cases}
\end{align*}
\]

Set up, but do not solve, the Wiener-Hopf equation for the case of a three-tap filter. If the matrix \( R \) is singular or numerically ill-conditioned, outline one method of obtaining a solution. [10 marks]
Indicative Solution for Question 4.

Answers to Question 5

(a) [12 marks] Figure 3 and the following.


The architectures for i), ii) and iii), respectively, are given in Figures 2, 3 and 4.

- For noise cancellation use the architecture. If we can obtain a reference signal $x[n]$ of the noise, the adaptive filter should be able to cancel the noise component from $e[n]$, provided that the noise in $x[n]$ and $d[n]$ are linearly related.

- System identification: to identify an unknown LTI system with impulse response $c[n]$

- For inverse system identification/equalization, place the unknown system and the adaptive filter in series.

(b) [8 marks]

- Formulation of error signal $e[n]$:

$$e[n] = d[n] - y[n] = d[n] - \sum_{i=0}^{L-1} w_i x[n-i] = d[n] - w^H x_n$$

Figure 3
with
\[ w = [w_0, w_1, \ldots, w_{L-1}]^H; \]
\[ x_n = [x[n], x[n-1], \ldots, x[n-L+1]]^H \]  

Mean Square Error formulation (note that the optimum filter \( w_{\text{opt}} \) is deterministic!):
\[ \varepsilon\{e^2[n]\} = \varepsilon\{(d[n] - w^T x_n)^2\} \]  
\[ = \varepsilon\{d^2[n]\} - 2w^T \varepsilon\{d[n]x_n\} + w^T \varepsilon\{x_n x_n^T\} w \]  
\[ = \sigma_d^2 - 2w^T p + w^T R w \]  

Given the auto-correlation sequence \( r_{xx}[\tau] \) and the cross-correlation sequence \( r_{xd}[\tau] \):
\[ r_{xx}[\tau] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot x[n-\tau] \]
\[ r_{xd}[\tau] = \frac{1}{N} \sum_{n=0}^{N-1} d[n] \cdot x[n-\tau] \]

The covariance matrix \( R \) and cross-correlation vector \( p \) can be written
\[
R = \begin{bmatrix}
r_{xx}[0] & r_{xx}[1] & \cdots & r_{xx}[L-1] \\
r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[L-2] \\
\vdots & \vdots & \ddots & \vdots \\
r_{xx}[L-1] & r_{xx}[L-2] & \cdots & r_{xx}[0]
\end{bmatrix},
p = \begin{bmatrix}
r_{xd}[0] \\
r_{xd}[1] \\
\vdots \\
r_{xd}[L-1]
\end{bmatrix}
\]

(c) [10 marks]
\[
R = \begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix},
p = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

If ill-conditioning arises AWGN is one solution.