Problem 1

1. Determine the auto-correlation function for the particular case when the input signal is given by

\[ x[n] = \cos n\pi \]

Hence show that the Wiener-Hopf solution for a 2-tap filter is not viable in this case.

2. Suppose that the input signal \( x[n] \) in the last part is changed to one whose auto-correlation is given by

\[ r_{xx}[\tau] = (-1)^\tau + \sigma^2 \delta(\tau) \]

(where \( \delta(\tau) \) is the Dirac impulse function). Does the Wiener-Hopf solution now exist and if so why?
Problem 1 Solution (first part)

Autocorrelation function

\[
 r_{xx}[\tau] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x[n] x^*[n - \tau] 
\]

For \( x[n] = \cos n\pi \) (cosine at 1/2 the sampling rate),

\[
 r_{xx}[\tau] = (-1)^\tau 
\]

The corresponding covariance matrix is

\[
 R = \begin{bmatrix}
 1 & -1 \\
 -1 & 1 
\end{bmatrix}
\]

which has rank equal to one and is not invertible. Hence the Wiener-Hopf solution does not exist.
Auto-correlation function in this case is

\[ r_{xx}[\tau] = (-1)^{\tau} + \sigma^2 \delta[\tau] \]

Hence

\[ R = \begin{bmatrix} 1 + \sigma^2 & -1 \\ -1 & 1 + \sigma^2 \end{bmatrix} \]

which is invertible for \( \sigma \neq 0 \) and the Wiener-Hopf solution exists.
Problem 2

In a particular application of Winer-Hopf filtering, it is known that

\[ \epsilon \{x[n]x[n - \tau]\} = \begin{cases} 
  b, & \tau = -1 \\
  a, & \tau = 0 \\
  b, & \tau = 1 \\
  0, & \text{elsewhere}
\end{cases} \]

and
Problem 2

\[ \epsilon \{ d[n] x[n - \tau] \} = \begin{cases} 
1, & \tau = -1 \\
c, & \tau = 0 \\
d, & \tau = 1 \\
e, & \tau = 2 \\
f, & \tau = 3 \\
0, & \text{elsewhere.} 
\] 

Compute \( w_{opt} \) for this case when a 2-tap filter is to be used.
In this case $L = 2$ and

$$R = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

and

$$p = \begin{bmatrix} r_{xd}[0] \\ r_{xd}[1] \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

Wiener-Hopf solution only when $a \neq b$. 
Problem 3 a) (January 2013 Exam Paper Q5)

In an application it is required to estimate a signal $y[n]$ from

$$x[n] = y[n] + v[n]$$

where $y[n]$ and $v[n]$ are uncorrelated. Also $x[n]$ is a wide-sense stationary process for which the first three values of its autocorrelation sequence are 2.5, 0 and 1 and the first four values of the cross correlation between $x[n]$ and $y[n]$ are 1.5, 0, 1 and 1.

Set up the Wiener-Hopf equation for the causal filter

$$W(z) = w[0] + w[1]z^{-1} + w[2]z^{-2}$$

and confirm that the solution is

$$[ w[0] \ w[1] \ w[2] ]^T = [ 0.5238 \ 0 \ 0.1905 ]^T$$
Problem 3 b) (January 2013 Exam Paper Q5)

Repeat the design of Problem 3 a) for the noncausal filter

\[ W(z) = w[-1]z + w[0] + w[1]z^{-1} \]

and confirm that the solution is

\[
\begin{bmatrix}
  w[-1] & w[0] & w[1]
\end{bmatrix}^T =
\begin{bmatrix}
  0 & 0.6 & 0
\end{bmatrix}^T
\]
Problem 3a) Solution

Wiener-Hopf equation

\[
\begin{bmatrix}
2.5 & 0 & 1 \\
0 & 1.5 & 0 \\
1 & 0 & 1.5
\end{bmatrix}
\begin{bmatrix}
w[0] \\
w[1] \\
w[2]
\end{bmatrix}
= \begin{bmatrix}
1.5 \\
0 \\
1
\end{bmatrix}
\]

Direct substitution for the given values shows that this equation is satisfied.
Problem 3b) Solution

Wiener-Hopf equation

\[
\begin{bmatrix}
2.5 & 0 & 1 \\
0 & 1.5 & 0 \\
1 & 0 & 1.5 \\
\end{bmatrix}
\begin{bmatrix}
w[-1] \\
w[0] \\
w[1] \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
1.5 \\
0 \\
\end{bmatrix}
\]

Direct substitution for the given values shows that this equation is satisfied.