The dynamics of molecules

The basic structure of biological material is comprised of a range of different molecules which exist in an environment in which they interact or react with other molecules or combine to form large structures.

Biomolecules generally have complicated structures which are related to their ability to carry information (in the case of DNA) or act as a structure or perform mechanical operations (in the case of proteins) and so on.

The behaviour of biological systems also critically depends on one of the most fundamental and smallest molecules, that of water, and also on single charged atoms (ions) which exist in high concentrations.

The natural movement of these molecules in response to stimuli is integral to the operation of life.

As a result, an understanding of the basic physics which affects the behaviour of molecules in solution is essential to understanding bionano systems and nanoscale systems in general.

Physics at small scales is dominated by Brownian motion, the random movement caused by the thermal or internal energy of matter. The net or average effect of Brownian motion over millions of molecules or ions is referred to as natural diffusion and governs, for example, the rates of chemical reactions and limits the motion of molecules. This is also the basis of osmosis, the mechanism by which biological cells control the motion of ions, water and food through the membrane.
Brownian Motion

In 1827, a botanist, Robert Brown, reported on the behaviour of pollen grains floating on water observed using a microscope. The grains exhibited an erratic behaviour, appearing to move randomly over time.

Brown theorised that the grains were subject to repeated random impacts from the much smaller water molecules. Conceptually, the much smaller water molecules vibrate randomly and collide with the grain particle. The time average of all these impacts is zero but at any point in time, the cumulative instantaneous average of the impacts produces a net force on the grain. At a different time, this instantaneous average is in a different random direction, producing a random force over time on the grain.

This effect was termed Brownian motion. The theory describing the effect took longer to emerge, being the subject of one of Einstein’s landmark papers in 1905 and was also worked on by Langevin, who also published in 1905 and Schmoluchowski.
Brownian motion

One obvious assumption in this case is that the molecule/particle is much larger than the molecules that make up the fluid. This approximation is remarkably accurate even for small molecules that would apparently violate the assumption. However, it is still an assumption based on an object immersed in another material.

In a gas, the continual movement of the molecules in random directions results in individual random collisions between molecules and therefore random motion. This is a distinct phenomena from that of a particle in a fluid.

However, the theoretical description of both cases is basically the same and is based on the concept of the random walk (a stochastic process).

Stochastic movement

Brownian motion can be illustrated in one dimension as a walker on a line randomly taking steps of equal length $l$ in either direction. After a time $t$, this walker has taken $N$ steps in total and his position $x(t)$ from the origin is equal to the sum of the $N$ steps taken:

$$x(t) = \sum_{i=1}^{N} l_i$$

where $l_i$ is a random selection of two values: $+l$ and $-l$. Statistically determining where the walker is likely to be after a given period of time can be indicated by determining the square of the displacement of the walker from the origin:

$$x'(t) = \sum_{i=1}^{N} l_i^2 = N[l^2]$$

Since the number of steps will be proportional in some way to time:

$$x'(t) = Dt$$

where $D$ is a diffusion coefficient. The instantaneous displacements for the walker grows over time but the average position remains at the origin and the distribution is Gaussian.

This simple model illustrates the idea of a stochastic or random process where the distribution of positions of a molecule does not depend on the starting time, only on the period over which the molecule is observed; it retains no "memory" of where it has been.

This last point is counterintuitive but important in that it is related to the consistency of the normal distribution.
Brownian motion

The random motion of atoms or molecules is related to the thermal or internal energy of the material. The kinetic theory of gases: the thermal energy or temperature is a measure of or equivalent to the kinetic energy stored in the molecules that comprise the gas.

From Statistical Mechanics, the equipartition of energy theorem gives the relationship between the temperature and the kinetic energy of each degree of freedom for the molecule:

\[ \frac{1}{2} m v^2 = \frac{1}{2} kT \]

For one degree of freedom

\[ \frac{1}{2} m v^2 = \frac{1}{2} 3kT \]

For three degrees of freedom

In a gas, the continual movement of the molecules in random directions results in random collisions between molecules and therefore random motion.

Brownian motion

For a molecule or particle in a dense fluid, the situation is different. Here we consider the molecule/particle of interest to be much larger than the molecules that make up the fluid. This approximation is remarkably accurate even for small molecules that violate the assumption.

In this case, we consider the stochastic equation of motion referred to as the Langevin equation:

\[ \frac{d\mathbf{v}}{dt} = -F_{\text{rand}}(t) - F_{\text{drag}}(t) \]

where \( F_{\text{rand}}(t) \) is the fluctuating force exerted on the particle by the fluid. Importantly, we consider that it can be divided into two parts:

1. The random stochastic force \( F_{\text{rand}}(t) \) with average value zero representing the numerous impacts of the molecules of the fluid on the particle.

2. The viscous drag force, generally written as the negative of the friction factor \( f \) times the velocity of the particle. The viscous drag of course, also arises from the impact of the molecules of the fluid on the particle, this time due to the motion of the particle. The momentum change on the leading face is greater than the trailing face, the net effect of which is a reactionary force slowing the particle.
Brownian Motion

The Langevin equation can be written as:

$$\frac{dx}{dt} = \frac{f}{m} + \frac{F_{\text{ext}}(t)}{m}$$

this equation can be solved by the following method:

multiply by $s$ and re-write in terms of $s$:

$$s \frac{dx}{dt} = \frac{fs}{m} + s \frac{F_{\text{ext}}(t)}{m}$$

given that

$$\frac{d}{dt} x^2 - 2 \frac{dx}{dt} v = \frac{d}{dt} x^2$$

we get

$$\frac{s}{2} \frac{d^2 x}{dt^2} = 1 \frac{d^2 x}{dt^2} v = \frac{d^2 x}{dt^2}$$

Brownian motion

The second term on the right is a transient term associated with the transition from the accelerating regime to the terminal velocity regime, where the viscous drag balances the applied force. This transient is characterised by the time constant $\frac{m}{f}$, often referred to as the momentum relaxation time.

We are only interested in the time average over long periods of time and the transient can be neglected:

$$\left\langle x^2 \right\rangle = 2kT \frac{m}{f}$$

per degree of freedom. In three dimensions, this is

$$\left\langle x^2 \right\rangle = 6kT \frac{m}{f}$$
Points to note

The friction factor for a spherical particle is (from the Stokes equation):

\[ f = 6 \pi r \eta \]

giving, from Einstein \( \left\langle x^2 \right\rangle = \frac{kT}{3 \pi \eta r} \) in one, or \( \left\langle x^3 \right\rangle = \frac{kT}{2 \pi \eta r} \) in three dimensions which was the expression given originally by Einstein.

This is generally referred to as the Stokes-Einstein formulation.

Brownian motion

Order of magnitude estimation of the relative importance of gravity and Brownian motion can be done by comparison with the buoyancy/sedimentation force

![Diagram showing displacement vs. particle radius]

Diffusion
Diffusion

Brownian motion is the stochastic movement of a single particle. The movement of an ensemble of many particles can be represented statistically or by considering diffusion of some ensemble property such as concentration or number density.

Diffusion was first derived phenomenologically from observations by Fick around 1855, years before the nature of Brownian motion was described.

When gases or fluids were observed moving into an empty space, the velocity at which the front moved was proportional to the gradient of concentration. This can be equally applied to the movement of a suspension of molecules.

Note: the equations equally apply to concentration, usually denoted \( c \), or number density, usually denoted \( n \). The difference between these two quantities is the dimensionless Avogadro’s number \( N_A \), as will be discussed later in more detail.

Fick’s First Law

Fick’s First Law applies for steady state diffusion, where the concentration gradient does not change with time in the volume of interest:

\[
J = -D \frac{\partial c}{\partial x}
\]

- \( J \) - Diffusive flux in terms of concentration (moles m\(^{-2}\) s\(^{-1}\))
- \( D \) - Diffusion coefficient (m\(^2\) s\(^{-1}\))
- \( c \) - Concentration (moles m\(^{-3}\))

This gives the relationship between flux due to diffusion and the gradient in concentration.

The Diffusion Equation can be derived from Fick’s First law.

Simple derivation of the diffusion equation

Flux from high concentration to low concentration along tube with fixed cross-sectional area through a fixed volume.
**Simple derivation of the diffusion equation**

Flux entering and leaving volume $\Delta V$:

$J_w$  

Amount entering and leaving:

$J_w \Delta V$  

Rate of change of concentration:

$$\frac{d\Delta N}{dt} = J_w \Delta V - J_i \Delta V$$

From Fick’s First Law:

$$J_w - J_i = -D \frac{\partial c}{\partial x} + D \frac{\partial^2 c}{\partial x^2} = -D \frac{\partial c}{\partial x} + D \frac{\partial^2 c}{\partial x^2} \left( c + \frac{\Delta V}{\Delta x} \right)$$

Substituting:

$$\frac{\partial^2 c}{\partial x^2} = \frac{D \Delta V}{\Delta x^2}$$

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**The diffusion equation**

Properly, the conservation of mass equation is used to derive Fick’s Second Law or the Diffusion Equation for time dependent concentrations.

Conservation of mass:

$$\frac{\partial c}{\partial t} + \nabla \cdot J = 0$$

Substitute

$$J = -D \frac{\partial c}{\partial x}$$

The diffusion equation:

$$\frac{\partial c}{\partial t} + \frac{\partial^2 c}{\partial x^2} = D \frac{\partial^2 c}{\partial x^2}$$

The diffusion equation is widely applicable in many areas, including thermal, electrical and molecular dynamics.

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**The value of the Diffusion Coefficient**

The relationship between Brownian motion and diffusion for molecules or ions can be demonstrated in several ways. As has been suggested by the use of the term diffusion constant in the derivation of Brownian motion, the determination of this constant is connected. Here, we examine two approaches:

- Forwards, from Einstein’s original paper on Brownian motion
- Backwards, from Kittel (Statistical Physics, Wiley).

Einstein’s discussion on diffusion begins with the consideration of the diffusing molecules under the following assumptions:

- one dimensional system: cylinder, cross-sectional area = 1
- sufficiently dilute that the motion of the molecules is due solely to the random thermal force and not due to intermolecular forces or collisions
Based on Brownian motion, we know that the positions of the molecules at time \( t \) do not depend on their previous states. After time \( \Delta t \), the positions of the molecules will have changed according to the random forces described previously. We know that the average of the positions is zero and that approximately half of the molecules will travel to the left and half to the right.

We assume that for time \( \Delta t \), for any point in the system, on average half of the particles experience the RMS displacement to the left and half to the right.

The value of the Diffusion Coefficient

During the time \( \Delta t \), the molecules that pass from left to right across \( S_r \) are those moving to the right within a distance \( \Delta x \) of \( S_r \), i.e., half of those in the layer \( S_r - S_{r-} \). The amount of molecules passing in this manner is therefore:

\[
\frac{1}{2} c \Delta L
\]

where \( c \) is the average concentration in the layer \( S_r - S_{r-} \), which is approximately equal to the concentration at \( P - P_{r+} \).

Similarly, the molecules passing left from layer \( S_{l-} S_{l+} \) is:

\[
\frac{1}{2} c \Delta L
\]

The net amount moving (diffusing) from left to right is therefore:

\[
\frac{1}{2} \Delta t (c_s - c_r)
\]
We can look at the concentrations around $S_i$ a different way. $c_-$ and $c_+$ are the concentrations in two planes, separated by a small distance $\Delta x$.

$$\frac{dc}{dx} \Rightarrow c_+ - c_- = -\Delta t \frac{dc}{dx}$$

Substituting, the net amount moving (diffusing) from left to right is

$$\frac{1}{2} \Delta t \frac{dc}{dx}$$

The amount moving (diffusing) from left to right per unit time (flux) is

$$\frac{1}{2} \Delta t \frac{dc}{dx}$$

Giving an expression for the Diffusion Coefficient:

$$D = \frac{1}{2} \Delta t \frac{dc}{dx}$$

Rearranging this gives:

$$\Delta x^2 = 2Da$$

Comparing with the value for Brownian motion displacement:

$$\langle x^2 \rangle = \frac{kT}{f}$$

gives the diffusion coefficient for the molecule

$$D = \frac{kT}{f}$$

where $f$ is the friction factor for the molecule
From Kittel, using the one dimensional diffusion equation
\[ \frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} \]
suppose at time \( t = 0 \), in an infinite domain, we have \( N \) particles concentrated at \( x = 0 \), the concentration is given by:
\[ c(x, 0) = \frac{N}{\sqrt{4\pi D t}} \delta(x) \]
where \( \delta(x) \) is the Dirac delta function. The solution of the diffusion equation for times greater than zero is:
\[ c(x, t) = \frac{N}{\sqrt{4\pi D t}} \exp \left( -\frac{x^2}{4D t} \right) \]
integrating over \( x \), gives
\[ \Delta x^2 = 2DAt \]

Summary

The relationship between Brownian motion and diffusion is important. Individual molecules in a cell or a biological system behave according to the former in small numbers and the latter when the number is sufficiently large.

Molecules and ions which are present in large numbers represent a source of pressure and force on objects which can be significant if equilibrium is disturbed – for example by a concentration imbalance across a membrane.

Electric fields and Charges
Electric fields

The combination of electric fields (DC) and molecules, particularly in suspension is central to biological systems.

- Reactions between different molecules are guided by the electrostatic force.
- The behaviour of cell membranes and the proteins contained in them are governed by the transmembrane potential, which in turn is controlled by the different concentrations of ions inside and outside the cell.
- Molecular separation is commonly performed using differential charge in an applied DC electric field, a technique called electrophoresis.
- The measurement of the response to an applied DC electric field can be used to obtain reaction constants.

Electrical force on a particle: electrophoresis

The electrostatic force on a particle is given by the Coulomb force on the net charge of the particle. Biological particles (cells) generally have a finite surface charge density (usually negative, due to the presence of acid groups on the surface) and observation of the movement of particles in a uniform electric field is used both to characterise and also to separate particles.

\[
F_{\text{elec}} = QE = \int_{S} \sigma \, dS \, E
\]

As discussed previously, using the Langevin equation of motion

\[
\frac{d\mathbf{v}}{dt} = -\mathbf{F}_d + \frac{Q}{m} \mathbf{E}
\]

gives the following equation for the particle velocity

\[
\frac{dv}{dt} = \frac{f}{m} + \frac{Q E}{m}
\]

The solution of this equation is

\[
v = \frac{Q E}{f \left( 1-e^{-\frac{f t}{m}} \right)}
\]

where \( m f \) is the momentum relaxation time, describing the characteristic length of the acceleration phase of the motion.

\[
v = \frac{Q E}{m \ln(2)}
\]

The part of the expression in front of the electric field is referred to as the electrophoretic mobility.

Many particles, including biological particles, when suspended in an electrolyte, are surrounded by an electrical double layer (discussed later) consisting of counter-ionic charges. This screens the particle surface charge, so that the force on the particle has a different origin.
Electrodiffusion and ions

In biological systems, the suspending fluid is invariably an aqueous solution of ions referred to as an electrolyte. The definition of an electrolyte is a solution of free ions which behaves as an electrically conductive media. As stated above, the behaviour of these ions and their response to natural and applied electric fields is central to the operation of biosystems and the methods used to measure them.

The amount and type of ion in solution governs the conductivity and the molecular composition of the suspending medium governs the permittivity. The mechanics of conduction in an electrolyte are more complicated than for an ideal dielectric. The ions have a finite size, and their mobility in solution is affected by the high permittivity of the water and in turn, high concentrations of ions can affect the permittivity.

The behaviour of the ions in solution is governed first by Brownian Motion and diffusion, as discussed previously, and second by the movement in response to the electric field. This motion which is similar to electrophoresis is the drift component of what are commonly called drift-diffusion problems.

Ions and conductivity

The conductivity of an ionic solution is complicated since the current can be carried by many different types of ion, each with a different mobility. The current density for each ion of type $j$, with valence $z_j$ is

$$\mathbf{J}_j = z_j \mathbf{F} \mu_j$$

with the individual contribution given by

$$\sigma_j = \frac{z_j q \mu_j}{\mathbf{E}}$$

the ionic conductivity is

$$\mathbf{J}_i = z_i \mathbf{F} \mu_i$$

(Stimol$^{-1}$)

Ionic mobility and diffusion

The Einstein relation gives the mobility in terms of the diffusion coefficient

$$\mu_i = \frac{z_i D_i q}{kT} = \frac{z_i F}{kT} \frac{D_i}{\mathbf{F} \mu_i}$$

where, from before:

$$D_i = \frac{kT}{\mathbf{F} \mu_i}$$

$$D_i = \frac{kT}{6\pi \nu a_j} = \frac{qRT}{6\pi \mathbf{F} \nu a_j}$$

where $a_j$ is the radius of the ion.

Measurement of the mobility provides a method for measuring the Diffusion coefficient:

$$\mu_i = \frac{qD_i}{6\pi \nu a_j}$$
Ions in solution

An ion in aqueous solution is essentially a free charge placed in a dielectric with a high permittivity. The local electric field generated by the ions both orients and binds the water molecules close to the ion: the hydration shell.

The dynamics of the ion are therefore complicated by the formation of this shell. While the electrical force remains the same, the radius of the moving ion has to take into account the bound layer of water: the hydration radius.

A side note on “constants”

There are two sets of related constants in the last few slides related to each other:

\[ \frac{q}{kT} = \frac{F}{RT} \]

the "physics" ones and the "chemistry" ones

The standard definitions are as follows:

- The Faraday: \( F = N_A q = 9.6487 \times 10^4 \text{C mol}^{-1} \)
- The (molar) gas constant: \( R = N_A k = 8.3143 \text{JK}^{-1}\text{mol}^{-1} \)

Ions and conductivity

The total conductivity is given by sum of the contributions from each ion

\[ \sigma = \sum_j \Lambda_j \]

A typically used term here is the molar conductivity (Siemens mol\(^{-1}\)) \( \Lambda = \sigma / c \)

For higher concentrations, ion-ion interactions lead to higher order effects:

\[ \Lambda = \Lambda_{\text{free}} - K_c^{1/2} \]

As \( c \to 0 \), ions move freely without interaction and the molar conductivity tends to a constant value, referred to as the limiting molar conductivity

\[ \Lambda_{\text{lim}} = \hat{F}(\mu_+ + \mu_-) \]
**Ions and conductivity**

Example:

Simple electrolyte: K+Cl

- Symmetrical electrolyte: equal mobilities for potassium and chloride ions.
  - $\mu_+ = 7.62 \times 10^{-8}$ m$^2$V$^{-1}$s$^{-1}$ and $\mu_- = 7.91 \times 10^{-8}$ m$^2$V$^{-1}$s$^{-1}$
  
  $A_{\text{cell}} = 9.6485 \times 10^{-4} \times (7.62 \times 10^{-8} + 7.91 \times 10^{-8}) = 0.01498$ (S m$^{-2}$).

Noting that the correct definition for concentration requires multiplying by 1000, so that 1M KCl has a concentration of 1 mol litre$^{-1} = 1000$ mol m$^{-3}$.

Therefore 1 mM KCl has a conductivity of 14.98 mS m$^{-1}$ at 298K.

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**The Nernst potential**

The Nernst potential is an important concept for cell membranes, describing the equilibrium potential for a given ion across the membrane. This equilibrium potential is established by asymmetric and driven transport of ions and molecules across the membrane by channels. The natural electrical potential across the membrane is given by the sum of the equilibrium potentials for all ions.
Derivation of the Nernst potential

There are several methods for deriving the Nernst potential:

Consider the potential due to one type of ion diffusing across a single ion permeable membrane. As the ions diffuse, an electric field is set up by the redistribution of the net charge.

Derivation of the Nernst potential

Diffusive flux:

\[ J_{\text{diff}} = -D \frac{\partial c}{\partial x} \]

The diffusion sets up an electric field and a current that opposes the diffusion.

Conduction flux:

\[ J_{\text{cond}} = -\mu E \]

Conservation equation:

\[ D \frac{\partial c}{\partial x} = \mu \frac{\partial E}{\partial x} \]

\[ \varepsilon \frac{\partial E}{\partial x} = \frac{K T}{z q} \frac{\partial c}{\partial x} \]

\[ \varepsilon E = \frac{K T}{z q} \frac{\partial c}{\partial x} \]

\[ \varepsilon E = \frac{K T}{z q} \frac{\partial c}{\partial x} \]

Potential difference across membrane

\[ -\Delta V = \int_{c_{\text{inside}}}^{c_{\text{outside}}} \gamma E dx = \frac{K T}{z q} \int_{c_{\text{inside}}}^{c_{\text{outside}}} \frac{dc}{c} \]

\[ \Delta V = -\frac{K T}{z q} \left( \ln(c_{\text{outside}}) - \ln(c_{\text{inside}}) \right) \]

Giving the Nernst equation

\[ \Delta V = \frac{K T}{z q} \ln \left( \frac{c_{\text{outside}}}{c_{\text{inside}}} \right) \]

The Nernst Potential for an ion across a cellular membrane is the potential which balances and opposes the diffusion of that ion through the membrane. Also referred to as the equilibrium potential or the reversal potential.
Alternative derivation

Boltzmann equation:

\[ P_i = e^{\frac{E_i - E_f}{RT}} \]

difference in energy probabilities

In a more relevant form

\[ \frac{c_i}{c_1} = e^{\frac{u_i - u_f}{RT}} \]

concentration or "activity"

This gives:

\[ u_i - u_f = RT \ln \left( \frac{c_i}{c_1} \right) \]

This is difference in molar electrical energy due to a potential difference:

\[ u_i = u_f = zF \Delta V \]

Giving:

\[ \Delta V = \frac{RT}{zF} \ln \left( \frac{c_i}{c_1} \right) = \frac{kT}{zq} \ln \left( \frac{c_i}{c_1} \right) \]

Example: Proton Motive Force

The proton-motive force is an example of a Nernst-like equation, where the chemical potential of the gradient in protons produces a mechanical output. It is defined as:

\[ \text{pmf} = \Delta V + 2.3026 \frac{kT}{q} \ln \left( \frac{\text{pH}_{1}}{\text{pH}_{2}} \right) \]

Example: membrane separating 180mM K+ from 500mM K+ and a pH difference of 7 to 5.5

\[ \Delta V = -\frac{kT}{zq} \ln \left( \frac{c_{\text{out}}}{c_{\text{in}}} \right) = 1.38 \times 10^{-23} \times 293 \ln \left( \frac{500}{180} \right) = 25.8 \text{mV} \]

\[ 2.3026 \frac{kT}{q} \ln \left( \frac{\text{pH}_{1}}{\text{pH}_{2}} \right) = 2.3026 \frac{1.38 \times 10^{-23} \times 293 \ln \left( \frac{7}{5.5} \right)}{6.1} = 6.1 \text{mV} \]

Example: Donnan Equilibrium

(Thermodynamic equilibrium in the cellular membrane)

Cell membranes tend to be permeable to both potassium and chloride. In cells, the presence of large proteins (negative ions) inside the cell, stabilises a gradient in K+ and Cl− across the membrane. This is referred to as Donnan Equilibrium:

\[ \ln \left( \frac{c_{\text{out}}}{c_{\text{in}}} \right) = \ln \left( \frac{c_{\text{out}}}{c_{\text{in}}} \right) \]

Potassium concentrations

\[ K_{\text{in}} = \frac{C_{\text{in}}}{K_{\text{out}}} \]

Chloride concentrations

\[ K_{\text{out}} = \frac{C_{\text{out}}}{K_{\text{in}}} \]

\[ K_{\text{in}} \times C_{\text{in}} = K_{\text{out}} \times C_{\text{out}} \]
Example: The Goldman Equation
(Resting potential of the cellular membrane)

In neurons, sodium also plays a key role in the membrane but the membrane has a low permeability to sodium compared to potassium. Both positive ions need to be considered:

\[
\Delta V = -\frac{kT}{q} \ln \left( \frac{P_{K_{\text{out}}} K_{\text{out}} + P_{Na_{\text{out}}} Na_{\text{out}}}{P_{Na_{\text{in}}} Na_{\text{in}} + P_{K_{\text{in}}} K_{\text{in}}} \right)
\]

Including Chloride gives the overall potential (form of the Goldman equation):

\[
\Delta V = -\frac{kT}{q} \ln \left( \frac{P_{K_{\text{out}}} K_{\text{out}} + P_{Na_{\text{out}}} Na_{\text{out}} + P_{Cl_{\text{out}}} Cl_{\text{out}}}{P_{Na_{\text{in}}} Na_{\text{in}} + P_{K_{\text{in}}} K_{\text{in}} + P_{Cl_{\text{in}}} Cl_{\text{in}}} \right)
\]

Particle-particle interaction, surfaces, van der Waals force and the DLVO theory

van der Waals forces

A set of forces which are electrostatic in origin since they involve the action of charges on each other. Also referred to as Hamaker forces, they can be important in controlling the behaviour of colloidal particles (molecules) in suspension, their interaction with each other and with surfaces.

Typically, the van der Waals force between two particles is attractive and is considered a long-range effect, although long range is relative.

The van der Waals force is comprised of three forces which can be represented independently and summed to give the total effect:

- The Keesom interaction force
- Debye interaction
- The London dispersion force

The relative magnitudes of each depend on the type of atom/molecule involved.
The Keesom interaction force arises from the average interaction between two permanent dipoles. This type of interaction is both attractive and orientational. Water is a good example of a highly polar molecule in which this phenomena occurs. Brownian motion randomises the position and orientation of the particles but the Keesom interaction force reduces the free energy of the system through dipole-dipole interaction.

The Debye or induction interaction arises from the interaction between a permanent dipole (polar molecule) and the dipole it induces in a non-polar molecule. This short range interaction arises due to the electric field from the permanent dipole polarising the second molecule. This polarisation exerts the second molecule results in an induced dipole. This interaction is generally weak and not sufficient to produce orientation of the molecules.

The London dispersion force is the most significant force. This attractive force is always present and acts between all atoms or molecules, whether they are charged or uncharged. It is produced by random fluctuations in the distribution of the electron clouds around the atoms, which on short time scales produce instantaneous dipole moments. This dipole rapidly and randomly changes in direction and magnitude with a zero time average following thermal statistical arguments similar to Brownian motion. However, instantaneously, the field from the dipole induces dipole moments in the electron cloud of neighbouring atoms. The dipole attracts the induced dipoles producing a non-zero time-averaged force between the atoms. The London dispersion is the most significant force and correct derivation requires Quantum Mechanics. However, following Israelachvili, we can examine a simplified model.

In the Bohr model of the atom, we have an electron orbiting the nucleus at the first Bohr radius \( a_0 \), where the Coulomb energy is equal to the first ionisation potential, \( I = 2 \alpha \nu \), where \( \hbar \) is Plank’s constant and \( \nu \) the orbiting frequency.

\[
P_1 = \alpha_1 q
\]

\( \) instantaneous dipole

\[
P_2 = \frac{P}{4 \pi \varepsilon_0 \omega}
\]

\( \) electric field

This polarises an identical atom of polarisability \( \alpha_2 \).

\[
\alpha_2 = 4 \pi \varepsilon_0 \omega a_0^3
\]

\( \) induced dipole

\[
P_2 = \alpha_2 E = \frac{P \alpha_2}{4 \pi \varepsilon_0 \omega}
\]

Average potential energy of interaction:

\[
U = \frac{-P \alpha_2 \omega}{4 \pi \varepsilon_0 \omega} = \frac{(\alpha_1 q)^2 \alpha_2}{(4 \pi \varepsilon_0 \omega)^2}
\]
The London dispersion force

More generally:

\[ U = \frac{1}{2} \left( \frac{r_1^2}{4 \pi \varepsilon_0} + \frac{r_2^2}{4 \pi \varepsilon_0} \right) \]

Substituting for the Bohr radius and using other expressions, the interaction energy can be written as

\[ U = \frac{2 \alpha^2 \hbar^2}{(4 \pi \varepsilon_0)^2} \left( \frac{1}{r^2} \right) = -\frac{C}{r} \]

This is an approximate expression for the dispersion interaction between two identical atoms. London's famous published expression contained a numerical factor

\[ U = -\frac{C}{r} = \frac{3 \alpha^2 \hbar^2}{4 (4 \pi \varepsilon_0)^2} \]

The important point to note is that the interaction energy falls as the inverse sixth power of distance, and also depends on the polarisability of the atoms. In order to calculate the interaction between large particles or surfaces, the force must be summed from all the individual atoms.

The Hamaker constant (Israelachvili 1992, Hamaker 1937), given by

\[ A = \frac{\lambda \omega}{6 \gamma y} \]

\[ A = \frac{\Lambda}{12 \gamma y} \]

\[ A = \frac{\Lambda}{12 \pi y} \]

These constants are calculated for vacuum, but can easily be modified for interactions in liquids or gases. The significance of this table lies in the differences in the distance-dependence of interaction energies.

For two interacting atoms the force falls off very quickly. However, for two colloidal particles the energy of interaction varies with the inverse of the distance \( y \), leading to a long-range interaction.

In order to calculate the force of interaction, the energy must be differentiated with respect to distance. The attractive force between two spheres is

\[ F = \frac{\Lambda \omega}{12 \pi y} \]

As an example, the force of attraction between two colloidal particles of 100 nm radius, with a separation of 1 nm is only 1 nN and for two cells of 5 \( \mu \)m radius, this is a not insignificant 50 nN.
Electrode-system interactions

The Electrical Double Layer

Static electrical potentials

Standard electrochemical interaction in the form of a measurement or a stimulus involves the application of an electrical potential. Note that this requires at least two electrodes.

A potential difference at a surface attracts ions in solution to form the Electrical Double Layer, a cloud of ions neutralising the charge on the surface.

The electrical potential in the bulk decreases exponentially from the surface in the steady state. This behaviour represents a balance between the electrostatic force attracting the ions towards the electrode and the diffusion generated by the resulting concentration gradient pushing the ions away.

This layer is sometimes referred to as the diffuse double layer and has a thickness $\lambda_D$, which is the characteristic decay length (or Debye length).
The Electrical Double Layer

The reciprocal Debye length is given by:

\[ \kappa = \frac{1}{\lambda_D} = \sqrt{\sum \frac{Z_i^2 \phi_i}{\epsilon kT}} \]

Or in terms of concentration (simplifying for symmetrical electrolytes) as:

\[ \kappa = \sqrt{\frac{2F^2c_0}{\epsilon kT}} \]

Two approximations

Debye-Hückel – thick double layer – small particle/molecule

\[ \phi = \phi_{TH} \]

Gouy-Chapman – thin double layer – flat surface

\[ \phi = \phi_{GC} = \frac{2kT}{q} \left( \frac{1 + \gamma e^{-\gamma}}{1 - \gamma e^{-\gamma}} \right) = \frac{4kT}{q} \gamma e^{-\gamma} \quad \gamma = \tanh \left( \frac{\phi_{GC}}{2kT} \right) \]

In both cases:

\[ \kappa^{-1} = \lambda_D = \sqrt{\frac{2F^2c_0}{\epsilon kT}} \]

Total energy of interaction: DLVO theory

The total interaction between two particles consists of two effects acting in opposition, the repulsive double layer interaction and the attractive van der Waals interaction. At any distance, the sum of these forces determines whether the interaction energy is positive or negative.

From our discussion on the Double layer, consider the interaction energy of two overlapping double layers. The repulsive force between two similarly-charged particles contains an electrostatic and an osmotic force component.

Consider two large particles with thin double layers coming together. As they approach, the number of countercations between the particles must remain constant to maintain electroneutrality. Bringing the surfaces closer and closer squashes the double layer ions into a smaller space (closer to the particles surface). This gives rise to an osmotic pressure from diffusion, which is proportional to the concentration of ions. The energy of repulsion can be found from the integral of the work done in bringing the particles together (from infinity) against the opposing pressure.
Total energy of interaction: DLVO theory

From Derjaguin, Landau, Verwey and Overbeek

Examples of the energy of interaction of Double Layers:

Two flat surfaces
Monovalent ions

\[ U_1 = \frac{64\pi n k T y_0}{\kappa} \]

\[ y = \tanh \left( \frac{2q\Phi}{kT} \right) \]

Two spheres of radius \( a \)

\[ U_2 = \frac{64\pi n k T y_0}{\kappa} \]

\[ U_3 = \frac{2\pi R e T y_0}{\kappa} \]

Force

\[ F_y = \frac{2\pi R e T y_0}{\kappa} e^{-\sigma} \]

Total energy of interaction: DLVO theory

The total energy of interaction is always negative (attractive force) when the surfaces are closer than a critical distance and the van der Waals force dominates and for large distances, the long range van der Waals forces dominate and again the force is attractive (negative potential).

Total energy of interaction: DLVO theory

For a high surface charge density and/or low electrolyte concentration, where a significant repulsive force exists, the surfaces strongly repel until their approach distance is less than the order of the Debye length.

Intermediate case where the force-distance curves show a secondary minimum. If this secondary potential is deep enough, particles can remain in solution without aggregating; they can adhere, but the process is reversible.

“Critical coagulation concentration”, particles adhere and then coagulate when the electrolyte is raised above a critical concentration.

If the surfaces have very low or zero surface charge.

(a) (b) (c) (d)
Electrode Potentials

The reaction is continuous and in equilibrium:

$$\text{Cu}^{2+} + 2e^- \rightarrow \text{Cu}$$

Each type of metal has a different equilibrium in the form of a concentration of ions produced. The equivalent potential for a measurement cell is the combination of the effects of multiple metals, determined using the Nernst equation.