Shape Registration

- What is shape registration?

- We would like to calculate the scaling, rotation and translation parameters to minimise the difference (distance) between two shapes.

- There is a lot of applications for this in medical imaging, Stereo Vision and etc.
- Shape Registration using Contours:

- Let us assume that shapes are represented by a set of points forming contours

- Each point has x and y coordinates, i.e.

\[ p_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \]

- A contour is then defined as the collection of its points, i.e.:

\[ C_1 = \begin{bmatrix} p_1^{c_1} & p_2^{c_1} & \ldots & p_N^{c_1} \end{bmatrix} \]

\[ C_2 = \begin{bmatrix} p_1^{c_2} & p_2^{c_2} & \ldots & p_N^{c_2} \end{bmatrix} \]
- A “distance” function between two contours is also defined:

\[ d(C_1, C_2) = \sum_i \left[ (x_i^{C_1} - x_i^{C_2})^2 + (y_i^{C_1} - y_i^{C_2})^2 \right] = |C_1 - C_2|^2 \]

- The translation, scaling and rotation parameters of a single point is considered to have the following form:

\[
Q(x, y) = \begin{bmatrix} x' \\ y' \end{bmatrix} = s \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}
\]

- The above, transforms point A to point A’ by scaling s, rotation \( \theta \), and translation \( (T_x, T_y) \).

- If every point on a contour is transformed by the above transformation, then the whole contour is transformed by scaling s, rotation \( \theta \), and translation \( (T_x, T_y) \).
In order to find the scaling, rotation and translation parameters, we need to minimise the distance function defined above, i.e.:

$$\min_{s,\theta,T_x,T_y} d\left(C_1, M_{s,\theta,(T_x,T_y)}C_2\right) = \min_{s,\theta,T_x,T_y} \left| C_1 - M_{s,\theta,(T_x,T_y)}C_2 \right|^2$$

The above distance function is non-linear with respect to $\theta$. The minimisation of the above term with respect to $\theta$ should be iterative and may fall in to local minima.

To avoid this, we suggest the following change of variables:

$$m = s \cos(\theta)$$
$$n = s \sin(\theta)$$

Therefore:

$$\min_{s,\theta,T_x,T_y} \left| C_1 - M_{s,\theta,(T_x,T_y)}C_2 \right|^2 = \min_{m,n,T_x,T_y} \left| C_1 - M_{m,n,(T_x,T_y)}C_2 \right|^2$$
\[
\min_{s, \theta, T_x, T_y} d\left(C_1, M_{s, \theta, (T_x, T_y)} C_2\right) = \min_{m, n, T_x, T_y} \left| C_1 - M_{m, n, (T_x, T_y)} C_2 \right|^2
\]

- The above distance function is linear with respect to all parameters \( m, n \) and \((T_x, T_y)\)
- Therefore we can simply calculate the parameters by using a linear least square scheme.

- Examples:

Two C3 vertebrae before registration

Two C3 vertebrae after registration
- Examples:
- Sign Distance functions and Registration:

- Let us consider the contour $S$. The Signed distance function is defined as:

$$\Phi_S(x, y) = \begin{cases} 
0 & (x, y) \in S \\
+ED((x, y), S) > 0 & (x, y) \in R_S \\
-ED((x, y), S) < 0 & (x, y) \in [\Omega - R_S]
\end{cases}$$
For registration between two shapes $D$ and $S$, we need to find the scaling, rotation and translation parameters minimising the following term:

$$E(s, \theta, T_x, T_y) = \iint_{\Omega} (s\phi_D(x, y) - \phi_S(Q(x, y)))^2 \, dx \, dy$$

- The optimal parameters can be calculated by taking the derivative of the above term with respect to the parameters, and setting it equal to zero.

$$\frac{dT_x}{dt} = 2\iiint_{\Omega} \frac{\partial \phi_S}{\partial x} (s\phi_D - \phi_S(Q)) \, dx \, dy$$

$$\frac{dT_y}{dt} = 2\iiint_{\Omega} \frac{\partial \phi_S}{\partial y} (s\phi_D - \phi_S(Q)) \, dx \, dy$$

$$\frac{d\theta}{dt} = 2\iiint_{\Omega} \left( \nabla \phi_S \cdot \frac{\partial \mathbf{A}}{\partial \theta} \right) (s\phi_D - \phi_S(Q)) \, dx \, dy$$

$$\frac{ds}{dt} = 2\iiint_{\Omega} \left( \phi_D + \nabla \phi_S \cdot \frac{\partial \mathbf{A}}{\partial s} \right) (s\phi_D - \phi_S(Q)) \, dx \, dy$$

$$\mathbf{A} = (sx \cos(\theta) + sy \sin(\theta)) \mathbf{i} + (-sx \sin(\theta) + sy \cos(\theta)) \mathbf{j}$$
- Example:

\[ s = 1.27, \ \theta = 71.38^\circ, \ T_x = -19.65, \ T_y = 21.32 \]
$s = 0.63$

$\theta = 59.94^\circ$

$T_x = -13.97$

$T_y = -14.09$

Real data

(1)

(2)
- **Advantages:**
  It can deal with complex shapes

- **Disadvantages:**
  It is iterative
  It may fall into local minima
  It may become unstable