Restoration

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What is it?

A picture is corrupted and we want to ‘clean’ it up

- Blur - out of focus, motion
- Noise - poor comms. (space probe), old video tape (scene of crime)
- Poor Technology - poor camera, limited bandwidth
Image Formation and Restoration

Forward Problem
(Image Formation)

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Inverse Problem

imaged data = f(‘real data’)

Hence

‘real data’ = f^{-1}(imaged data)

So-called inverse problem.

Given f, find f^{-1}.

This is a hard problem
Image Formation

Spatial Domain

“Real World” \( u(x_1, x_2) \) \( \rightarrow \) \( h(x_1, x_2) \) \( \rightarrow \) \( v(x_1, x_2) \) \( \rightarrow \) Image

Impulse response of imaging system

\[ v = h \ast u \]

Frequency Domain

“Real World” \( U(\omega_1, \omega_2) \) \( \rightarrow \) \( H(\omega_1, \omega_2) \) \( \rightarrow \) \( V(\omega_1, \omega_2) \) \( \rightarrow \) Image

Transfer function of imaging system

\[ V = H \ast U \]
 Imaging System

\[ u(x_1, x_2) \rightarrow h(x_1, x_2) \rightarrow v(x_1, x_2) \]

A linear system

\[ h(x_1, x_2) \] will not be a delta function if problems exist, e.g. out of focus, detector noise, lens non-linearity, ...
Image Restoration

The problem:

Given
\( v \) (the image) and
\( H \) (the transfer function of the imaging system)*
find
\( u \) (the ‘real world’ image)

* or an approximation to it
Inverse Filtering

\[ V = HU \]

\[ U = \frac{V}{H} \]

**Advantages**

- Straightforward approach

**Disadvantages**

- What happens if \( H \) is zero at some point?
Pseudo Inverse Filtering

\[ V = HU \]

\[ U = H^{-1}V \]

\[ H^{-1}(\omega_1, \omega_2) = \begin{cases} 
\frac{1}{H(\omega_1, \omega_2)} & H(\omega_1, \omega_2) \neq 0 \\
0 & H(\omega_1, \omega_2) = 0
\end{cases} \]

This means there will be some frequencies which we know nothing about
Transfer Function

How can we find $H$?
• Prior Knowledge of System
• Try different models and see which is best
• Measure it (use known Test image)
• Assume Gaussian (Central Limit Theorem)
• Deduce it as part of problem
Atmospheric Turbulence

\[ h(x_1, x_2) = e^{-\pi k^2 (x_1^2 + x_2^2)} \]

\[ H(\omega_1, \omega_2) = \frac{1}{k^2} e^{\frac{-\pi}{k^2} (\omega_1^2 + \omega_2^2)} \]
Blur due to uniform motion. Can think of this as averaging image intensity over an interval of size $d$. 
Image Formation with Additive Noise

**Spatial Domain**

```
“Real World”  \[ u(x_1, x_2) \] \[ h(x_1, x_2) \] \[ v(x_1, x_2) \] \[ \text{Image} \]
```

\[ v = h \ast u + n \]

\[ n(x_1, x_2) \] Noise

**Frequency Domain**

```
“Real World”  \[ U(\omega_1, \omega_2) \] \[ H(\omega_1, \omega_2) \] \[ V(\omega_1, \omega_2) \] \[ \text{Image} \]
```

\[ V = H \cdot U + N \]

\[ N(\omega_1, \omega_2) \] Noise
Pseudo Inverse Filtering

\[ V = HU + N \]

\[ U \approx \hat{U} = H^{-1}V - H^{-1}N \]

If \( H \) is small when \( N \) is non-zero then \( H^{-1}N \) can dominate
Wiener Filtering

**Motivation:**
Inverse and Pseudo-Inverse Filters are sensitive to Noise

**Goal:**
Develop a technique which can restore images in the presence of blur and noise

**Solution:**
Wiener Filtering
Wiener Filtering

Suppose our system is modelled by a linear observation system with additive noise

\[ v = h \ast u + n \quad V = HU + N \]

Wiener filtering

Inverse Filtering + Noise Suppression
Optimal Noise Suppression

\[ G_{\text{noisespn}}(\omega_1, \omega_2) = \frac{|S(\omega_1, \omega_2)|^2}{|S(\omega_1, \omega_2)|^2 + |M(\omega_1, \omega_2)|^2} \]

- \[ |S(\omega_1, \omega_2)|^2 \] \hspace{1cm} Power Spectrum of Signal
- \[ |M(\omega_1, \omega_2)|^2 \] \hspace{1cm} Power Spectrum of Noise

Quick Check, consider limits

- **No Signal** \[ G_{\text{noisespn}}(\omega_1, \omega_2) \to 0 \]
- **No Noise** \[ G_{\text{noisespn}}(\omega_1, \omega_2) \to 1 \]
1D Example

\[ |S(\omega)| \]

\[ |M(\omega)| \]

\[ G_{\text{noisespn}}(\omega) = \frac{|S(\omega)|^2}{|S(\omega)|^2 + |M(\omega)|^2} \]
Wiener Filter

Recover $\mathbf{U}$ (or good estimate) using a filter $G$

$$\mathbf{U} \approx \hat{\mathbf{U}} = \mathbf{GV} - \mathbf{GN}$$

Two parts to the filter -> Noise Suppression + Inverse Filter

$$G = H^{-1}G_{\text{noisespn}} \quad \text{where} \quad G_{\text{noisespn}} = \frac{|\mathbf{S}|^2}{|\mathbf{S}|^2 + |\mathbf{M}|^2}$$
Wiener Filtering

\[ V = H U + N \]

Inverse filtering

\[ \frac{V}{H} = \frac{U}{\text{signal}} + \frac{N}{\text{noise}} \]

Noise Suppression

\[ G_{\text{noise supn}} = \frac{|U|^2}{|U|^2 + \left| \frac{N}{H} \right|^2} \]

\[ G = H^{-1} \frac{|U|^2}{|U|^2 + \left| \frac{N}{H} \right|^2} = \frac{H^*}{|H|^2 + \left| \frac{N}{U} \right|^2} \]
Wiener Filter Derivation

- We can derive the wiener filter by minimising the mean square error between the original signal, \( x(t) \) and its estimate \( \hat{x}(t) \) to find the optimal filter \( g(t) \), i.e.:

\[
y(t) = h(t) * x(t) + n(t)
\]

\( y(t) = \) Observed signal

\( x(t) = \) Original Signal

\( h(t) = \) Impulse Response of a linear invariant system

\( n(t) = \) Noise

\( \hat{x}(t) = g(t) * y(t) = \) An estimate of \( x(t) \)
Our objective is to find $g(t)$ so that the mean square error between signal $x(t)$ and its estimate $\hat{x}(t)$ is minimum. We do this in frequency domain.

\[
\varepsilon(f) = E \left[ \left| X(f) - \hat{X}(f) \right|^2 \right] = E \left[ \left| X(f) - G(f)Y(f) \right|^2 \right]
\]

\[
\varepsilon(f) = E \left[ X(f) - G(f)\left[ H(f)X(f) + N(f) \right] \right]^2 \]

\[
\varepsilon(f) = E \left[ \left| 1 - G(f)H(f) \right| X(f) - G(f)N(f) \right|^2 \]

By expanding the quadratic, we can write:

\[
\varepsilon(f) = [1 - G(f)H(f)][1 - G(f)H(f)]^* E\left[ |X(f)|^2 \right] - [1 - G(f)H(f)]G^*(f)E\{X(f)N^*(f)\} - [1 - G(f)H(f)]^*G(f)E\{X^*(f)N(f)\} + G(f)G^*(f)E\left[ |N^*(f)|^2 \right]
\]

(1)
We assume that noise is independent of signal, therefore:

\[
E\{X(f)N^*(f)\} = E\{X^*(f)N(f)\} = 0
\]

\[
S(f) = E\|X(f)\|^2 = \text{Power Spectrum of Signal}
\]

\[
PN(f) = E\|N(f)\|^2 = \text{Power Spectrum of Noise}
\]

Therefore equation (1) becomes:

\[
\varepsilon(f) = \left[1 - G(f)H(f)\right]\left[1 - G(f)H(f)\right]^*S(f) + G(f)G^*(f)N(f)
\]

The minimum error value is obtained by differentiating with respect to \(G(f)\) and set equal to zero. \(G^*(f)\) is assumed to be constant.

\[
\frac{d\varepsilon(f)}{dG(f)} = G^*(f)N(f) - H(f)[1 - G(f)H(f)]^*S(f) = 0
\]

This then leads to the Wiener filter:

\[
G(f) = \frac{1}{H(f)} \left[ \frac{|H(f)|^2}{|H(f)|^2 + \frac{PN(f)}{S(f)}} \right]
\]
Filter Requirements

\[ G = \frac{H^*}{|H|^2 + |N/U|^2} \]

Need to know \( H, N, U \)

\( H \) as for inverse filtering

\( N \) is often known or can be approximated

\( U \) is usually not known and can be approximated by an image ensemble
Example
Examples

Original Image  Blurred Image  Restored Image
Relation to Inverse Filtering

Wiener filtering

\[ G = \frac{H^*}{|H|^2 + |\frac{N}{U}|^2} \]

Inverse filtering

\[ N = 0 \quad G = \frac{1}{H} \]

Pseudo-Inverse filtering

\[ \lim_{N \to 0} G = \text{pinv}(H) \]
Wiener Filter Example
- Total variation definition:

The total variation of a differentiable function $f$ defined on an interval $[a,b]$ has the following expression:

$$\text{Total Variation} := \int_{a}^{b} \left| \frac{df}{dx} \right| dx$$

The total variation for a differentiable function of several variables defined on a bounded open set $\Omega \subseteq R^n$ has the following expression:

$$\text{Total Variation} := \int_{\Omega} |\nabla f| dx^n$$

For 2D images, the total variation has the following form:

$$\text{Total Variation} := \iint_{\Omega} |\nabla f| dx dy$$
For denoising, we would like to find the smoothed image $u$ minimising the following term:

$$\text{minimise } \iint_{\Omega} |\nabla u| dx\,dy$$

Subject to:

$$\iint_{\Omega} \frac{1}{2} (u - u_0)^2 \, dx\,dy = \sigma^2$$

Where $\sigma^2$ is the variance of the noise and is assumed constant.

In the above, $u_0$ is the original noisy image.

Using Lagrange multipliers, we can write the following functional:

$$E = \iint_{\Omega} \left| \nabla u \right| + \lambda (u - u_0)^2 \, dx\,dy$$
We therefore arrive at the Euler-Lagrange equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \frac{u_x}{\sqrt{u_x^2 + u_y^2}} \right) + \frac{\partial}{\partial y} \left( \frac{u_y}{\sqrt{u_x^2 + u_y^2}} \right) - \lambda (u - u_0)$$

For \( t > 0 \)

- This is a nonlinear filter which preserves the discontinuities in the original image:

As \( \lambda \to +\infty \) \hspace{1cm} \text{There will be no denoising}

As \( \lambda \to 0 \) \hspace{1cm} \text{The final image (result) is constant everywhere}

- It is possible to denoise 1D signals with total variation as well. The corresponding functional has the following form:

$$E = \int_a^b \left[ \left| \frac{du}{dx} \right| + \lambda (u - u_0)^2 \right] dx$$