Lecture and Tutorial Slots

- Each Question worth 5 marks. Overall coursework mark is 10% of the final module mark.
- Marks have to be integers.
- Marks breakdown Question 1: [a] 2 marks, [b] 1 mark, [c] 2 marks.
- Marks breakdown Question 2: [a] 2 marks, [c] 3 marks.
Question 1 [a]

[a] A discrete linear system with input $x$ and output $y$ is described by the difference equation

$$y[n] = y[n - 1] + x[n] - x[n - 4]$$

Find the transfer-function of this system and calculate its poles and zeros. Compute also the impulse response of this system.
The transfer-function is obtained by taking the $z$-transform of the difference equation with zero initial conditions.

\[
\frac{Y(z)}{X(z)} = \frac{1 - z^{-4}}{1 - z^{-1}} = \frac{z^4 - 1}{z^3(z - 1)}
\]

The transfer-function is 4th order and has 4 poles and 4 zeros.

**Poles:** Three poles at zero and one pole at $z = 1$.

**Zeros:** $z^4 = 1$ and hence zeros at $z = \pm 1, \pm j$.

Hence there is a pole-zero cancellation at $z = 1$.

Hence the transfer-function is

\[
G(z) = z^{-3}(z + 1)(z + j)(z - j) = 1 + z^{-1} + z^{-2} + z^{-3}
\]
Question 1 [a] Answer cont’d

This is an FIR filter with impulse response \{1, 1, 1, 1\}.

Comments on the Answers Submitted
A significant number of answers failed to state that there are three poles at \( z = 0 \).
A system with more zeros than poles is non-causal and therefore physically not realisable.
A significant number of solutions did not recognise that the transfer-function is the ratio of two coprime polynomials, i.e., all common zeros and poles are cancelled.
These are fundamental mistakes so no marks awarded.
[b] Determine frequency response of the filter of the previous part of this question and the frequencies at which it is zero.

**Answer:**

\[
G(e^{j\omega}) = \frac{1 - e^{-j4\omega}}{1 - e^{-j\omega}}
\]

Frequency response is zero when \( \omega = \frac{n\pi}{2} \).
Question 1 [c]

A communications system operates at sampling rate of 800 kHz but suffers interference from a signal at 60 kHz. Design a second order IIR filter to remove this interference signal.

Answer:

Notch frequency in radian/sec

\[ \omega_0 = 2\pi \frac{60 \times 10^3}{800 \times 10^3} = \frac{6\pi}{40} \text{ rad/sec}. \]

Notch filter zeros required to be a complex conjugate pair

\[ z_0 = e^{j\omega_0} \text{ and } z_1 = e^{-j\omega_0} \]

This fixes the numerator of the filter transfer function as

\[ b(z) = (z - z_0)(z - z_1) \]
Next the stable pole locations must be selected.

\[ a(z) = z^2 \]

gives an **FIR filter**.

An **IIR filter** results from placing a pair of complex conjugate poles inside the unit circle, e.g.,

\[ a(z) = (z - b)(z - c) \]

with \(|b| < 1, |c| < 1\) corresponds to real poles.

Another solution is

\[ a(z) = (z - d_1 e^{j\omega_0})(z - d_1 e^{-j\omega_0}) \]

where \(d_1\) is real and \(|d_1| < 1\) (places the poles ‘close’ to the zeros and is popular in some applications).
A significant number of answers started by designing an analog filter and then converted to a discrete approximation using the bilinear transform. The bilinear transform preserves stability but frequency warping is an issue. In a notch filter it is essential to place the zeros at the frequency to be removed. Hence direct digital design is a superior method in this respect. Recall also that if you have an analog filter in pole-zero form then mapping the poles to the digital domain is straightforward but this is not the case for zeros.
Question 2 [a]

[a] For a particular application, the analogue filter $G(s)$ has the following properties in terms of the normalized frequency $\Omega$

i)\[1 - \gamma \leq |G(j\Omega)| \leq 1 + \gamma, \quad |\Omega| \leq \Omega_p\]

and ii)\[|G(j\Omega)| \leq \eta, \quad |\Omega| \geq \Omega_s\]

where $\gamma$ and $\eta$ are parameters describing the pass band ripple and stop band attenuation, respectively, $\Omega_p$ denotes the upper limit of the pass band and $\Omega_s$ is the lower limit of the stop band. In a particular application, a digital low pass filter $G(z)$ with pass band upper limit $\omega_p$ is to be derived from $G(s)$ using

$$G(z) = G(s)|_{s = \frac{2}{\beta} \left( \frac{z-1}{z+1} \right)}, \quad 0 \leq \beta \leq \infty$$

Find the relationship between $\beta$ and $\Omega_p$ to ensure that $\omega_p = \frac{\pi}{2}$. 
Question 2 [a] Answer

\[ s \mapsto \frac{2 (1 - z^{-1})}{\beta (1 + z^{-1})} \]

Frequency response \( z = e^{j\omega} \).
Also

\[ j\Omega_p = \frac{2 (1 - e^{-j\omega})}{\beta (1 + e^{-j\omega})} \]

Multiplying above and below by \( e^{j\omega/2} \) now gives

\[ \Omega_p = \frac{2}{\beta} \tan \frac{\omega_p}{2} \]

For \( \omega_p = \frac{\pi}{2} \)

\[ \beta = \frac{2}{\Omega_p} \]
Question 2 [b]

The signals $x[n], g[n], h[n], r[n]$ and $s[n]$, where $g[n]$ is a causal impulse response with an arbitrary phase characteristic, are known to satisfy the following equations where $\ast$ denotes convolution

\[
\begin{align*}
    h[n] &= g[n] \ast x[n] \\
    r[n] &= g[n] \ast h[-n] \\
    s[n] &= r[-n]
\end{align*}
\]

Find the DFT, $R(\omega)$, of the signal $r[n]$ in terms of $X(\omega)$ and $G(\omega)$. Find also the DFT, $S(\omega)$, of the signal $s[n]$ in terms of $X(\omega)$. Let

\[
G_1(\omega) = \frac{S(\omega)}{X(\omega)}
\]
Question 2 [b] cont’d

Find the gain and phase characteristics of $G_1(\omega)$ in terms of those for $G(\omega)$. If $G_1(\omega)$ is to be used as a filter, what property must the signal to be filtered have?
Question 2 [b] Answer

\[ H(\omega) = G(\omega)X(\omega) \]
\[ R(\omega) = G(\omega)H(-\omega) = X(-\omega)G(-\omega)G(\omega) \]
\[ S(\omega) = R(-\omega) = X(\omega)G(\omega)G(-\omega) \]

Hence
\[ \frac{S(\omega)}{X(\omega)} = G(\omega)G(-\omega) = G_1(\omega) \]

The impulse response is causal and real and hence
\[ |G(-\omega)| = |G(\omega)| \] and \[ \angle G(-\omega) = -\angle G(\omega). \]
Hence

\[ G(\omega)G(-\omega) = G_1(\omega) = |G(\omega)|e^{j\angle G(\omega)}|G(-\omega)|e^{j\angle G(-\omega)} = |G(\omega)|^2 \]

\( G_1(\omega) \) has magnitude \( |G(\omega)|^2 \) and zero phase. \( G_1(\omega) \) is the frequency response of a zero phase filter. To implement a zero phase filter all data must be available prior to filtering and hence time reversal is possible.