ELEC 6218 — Signal Processing — Discrete-time Signals (Basics)

Professor Eric Rogers

Department of Electronics and Computer Science
University of Southampton
etar@ecs.soton.ac.uk
Office: Building 1, Room 2041
Background

These signals, denoted by DT from this point onwards, are defined only at discrete time values. Usually the time values are equally spaced, i.e. at multiples of constant period $T$

$$t = n T, \ n \ \text{an integer}$$

Often DT signals are obtained by sampling continuous-time (CT) signals or waveforms, e.g. digitized input to a computer

$$x(n) = x(nT)$$

where here we follow the common practice of omitting $T$ from the argument. Note also that DT signals can be inherently discrete, e.g. daily takings in a shop.
Normalised Frequency

Consider the DT sinusoid

\[ n = \begin{array}{c}
-2 & -1 & 0 & 1 & 2 & 3 & 11 & 12 & 13 & 14 & 15 & 16 \\
\end{array} \]
Normalised Frequency

\[ x(n) = A \cos(\omega n T + \theta), \quad \omega = 2\pi f \]
\[ = A \cos(\Omega n + \theta), \quad \Omega = 2\pi F \]

where \( \Omega \) and \( F \) are normalized frequencies with respect to the sample rate \( f_s = \frac{1}{T} \), i.e.

\[ \Omega = \omega T = \frac{2\pi f}{f_s} \quad (\text{rad/sample}) \]
\[ F = f T = \frac{f}{f_s} \quad (\text{cycles/sample}) \]
Consider also the sinusoidal sequence $x_0(n) = A \cos(2\pi F_0 n)$ where the normalized frequency $F = F_0 < 0.5$, i.e. the actual frequency $f_0 < \frac{f_s}{2}$. Then it is clear from the next figure that a higher frequency CT sinusoid could be fitted to the same samples, e.g. with actual and normalized frequencies as given next.
Normalised Frequency

\[ f_1 = f_s - f_0 \]
\[ F_1 = 1 - F_0 \]

The samples of this higher frequency are given by

\[ x_1(n) = A \cos [2\pi(1 - F_0)n] \]
\[ = A \cos (2n\pi - 2n\pi F_0) \]
\[ = A \cos (2n\pi F_0) \]
\[ = x_0(n) \]

This phenomenon is known as aliasing and will be considered in detail later.
Power and Energy Signals

DT signals can be classified as power and energy signals in the same way as for CT signals, the only difference being in the definitions which become

\[ P = \lim_{N \to +\infty} \left[ \frac{1}{2N + 1} \sum_{n=-N}^{N} |x(n)|^2 \right] \]

and

\[ E = \sum_{n=-\infty}^{\infty} |x(n)|^2 < +\infty \]

respectively.

A DT signal is causal if

\[ x(n) = 0, \text{ for } n < 0 \]
Sampling Theory

Sampling is an essential process in digital control/signal/image processing. It imposes certain limitations on what can be achieved in subsequent processing and a thorough understanding of the principles behind it is essential. Two forms of idealized sampling of a CT signal will be considered here:

- natural
- instantaneous
Natural Sampling

This can be regarded as an **on/off switching operation** which can be modelled as multiplication of the CT signal by a train of unit amplitude rectangular impulses of width $\tau$ and period $T$, i.e.

$$x_s(t) = x(t)p(t)$$

where the **periodic pulse train** $p(t)$ is defined as

$$p(t) = \sum_{n=-\infty}^{\infty} \text{rect}(\frac{t - nT}{\tau})$$

Since $p(t)$ is periodic, it can be represented by its Fourier series (work the details)
Natural Sampling

Hence

\[ x_s(t) = x(t) \frac{\tau}{T} \sum_{n=-\infty}^{\infty} \frac{\sin \left( \frac{n\pi \tau}{T} \right)}{n\pi \tau} e^{jn\omega_s t} \]

\[ = \frac{\tau}{T} \sum_{n=-\infty}^{\infty} \text{sinc}(nf_s \tau) x(t) e^{jn\omega_s t} \]
Natural Sampling

Applying the Fourier transform (and making use of Property 4 in the notes on the Fourier transform) gives the spectrum of the sampled signal as

\[ X_S(f) = \frac{\tau}{T} \sum_{n=-\infty}^{\infty} \text{sinc}(nf_s \tau)X(f - nf_s) \]

This spectrum consists of the CT signal spectrum \( X(f) \) plus an infinite number of images of \( X(f) \) one centered on each multiple of the sampling frequency \( nf_s \) - see the next Figure (first diagram).
Natural Sampling

Instantaneous Sampling

(i) \( f_h < f_s / 2 \)

(ii) \( f_h > f_s / 2 \)
Instantaneous Sampling

In this form of idealised sampling (above Figure (last two diagrams)) the samples are represented as impulses with magnitudes (or areas) equal to the values of $x(t)$ at times $t = nT$, i.e.

$$x_S(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

or (by a basic property of the unit impulse function)

$$x_S(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = x(t)c(t)$$

where

$$c(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$
Instantaneous Sampling

This is a periodic train of unit impulses known as a comb function. This comb function can be described by its Fourier series

\[ c(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \]

Hence

\[ x_S(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} x(t)e^{jn\omega_s t} \]

and, on taking the Fourier transform, the spectrum of the sampled signal is given by

\[ X_S(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \]
Instantaneous Sampling

It is clear that

- all images of $X(f)$ are of equal magnitude
- if $X(f)$ is bandlimited such that the highest frequency $f_h < \frac{f_s}{2}$ then $X(f)$ can be recovered by using an ideal low pass filter (LPF) of the form (see also below)

$$H_i(f) = \text{rect}\left(\frac{f}{f_s}\right)$$

If, however, $f_h > \frac{f_s}{2}$ then the images of $X(f)$ overlap giving aliasing distortion.
Instantaneous Sampling

Consideration of these facts leads immediately to the following statement of the **Sampling Theorem** (Nyquist, Shannon). A CT signal which is **bandlimited** to the frequency range $\pm f_h$ is completely defined by taking samples at a uniform rate $f_s \geq 2f_h$

The **minimum sampling rate** $f_s = 2f_h$ is known as the **Nyquist rate**.

**Note:** In practical systems an **anti-aliasing low-pass filter** is used to restrict the CT signal bandwidth prior to sampling.
Interpolation

Here this term means **the recovery of the original CT signal by filling in the gaps between sampling** - a similar process to interpolation in numerical analysis (or even graph plotting). Three types of interpolation will be considered here — the first is purely ideal and the other two more practical.

**Ideal LP Filter**

We have already seen that in the frequency domain it is possible to recover the CT signal exactly using an ideal LP filter with transfer function

\[ H_i(f) = \text{rect}\left(\frac{f}{f_s}\right) = \text{rect}(fT) \]

and impulse response

\[ h_i(t) = \frac{1}{T}\text{sinc}\left(\frac{t}{T}\right) \]
Interpolation

Plot of this last expression
Interpolation

For instantaneous sampling it is straightforward to derive a time domain expression for the recovered CT signal, starting with the filter input sequence

\[ x_S(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \]

The filter output, denoted by \( x_r(t) \), then is simply the sum of the weighted and shifted impulse responses, i.e.

\[ x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)h_i(t - nT) \]

\[ = \frac{1}{T} \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\frac{t - nT}{T}\right) \]
Interpolation

\[
= f_s \sum_{n=-\infty}^{\infty} x(nT) \text{sinc}(f_s t - n)
\]

At the sampling points \((t = nT)\) all sinc functions are zero except the one centered at \(t = nT\). Hence the recovered signal is exact at these points, and provided the sampling theorem is satisfied, it is also exact at all intermediate values of \(t\), i.e.

\[
x_r(t) = f_s x(t)
\]
Interpolation

Illustration of the above process by showing the summation of the weighted and shifted impulse responses.

\[ x_r(t) = f_s x(t) \]
Interpolation

Illustration of the above process by showing the summation of the weighted and shifted impulse responses. For natural sampling, a similar analysis shows that the recovered signal is given by

\[ x_r(t) = \frac{T}{\tau} x(t) = \tau f_s x(t) \]

Non-Ideal LP Filter
Realisable LPFs can only approximate the ideal LPF response and will always have a finite response to the higher frequency spectral images giving distortion of the recovered CT signal. The next figure illustrates this fact.
Interpolation

non-ideal LPF

$X_s(f)$

$H(f)$

$f$

$0$  $f_s/2$  $f_s$  $2f_s$
Interpolation

Another minor problem with practical LPFs is that if $\frac{T}{\tau}$ is small then the filter output has a small amplitude and more gain is required.

**Zero Order Hold**

In this approach, each sample value is held constant until the next sample is input to the filter. This gives a staircase approximation of $x(t)$ as shown below. Clearly there is high frequency distortion due to the ‘sharp corners’ on the output waveform. There is also low frequency distortion as shown next.
Interpolation
Interpolation

For the case of instantaneous sampling, we can assume that the unit impulse response of the filter to be a unit amplitude rectangular pulse of duration $T$, i.e.

$$h_z(t) = \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

Hence on applying the Fourier transform, the effective transfer function of this filter is given by

$$H_z(f) = T \text{sinc}(fT)e^{-\frac{i\omega T}{2}}$$

where the exponential term here is equivalent to a delay of value $\frac{T}{2}$. 

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Interpolation

Plots of the gain $|H_z(f)|$ and the amplitude spectrum of the recovered CT signal.

Effective Frequency Response of Zero Order Hold