ELEC 6218 — Signal Processing — Digital Filter Design

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Digital Filter Design

- Digital filter $H(z)$.
- Frequency response $H(e^{j\theta}) = |H(e^{j\theta})|e^{-j\phi(\theta)}$.
- A continuous function of $\theta$ with period $2\pi$.
- Magnitude is an even function, i.e., $|H(e^{j\theta})| = |H(e^{-j\theta})|$.
- Phase is an odd function, i.e., $\phi(\theta) = -\phi(-\theta)$.
- **Assumption:** $H(z)$ only has real coefficients.
Digital Filter Design

- More convenient to use the **magnitude squared and the group delay functions**.

\[ |H(e^{j\theta})|^2 = H(z)H(z^{-1})|_{z=e^{j\theta}}. \]

- Group delay function \( \tau(\theta) = -\frac{d\phi(\theta)}{d\theta} \) — measure of the delay of the filter response.

- Complex poles and zeros occur in conjugate pairs.

- If \( z_k = a \) is a real pole/zero of \( |H(e^{j\theta})|^2 \) then \( z_k^{-1} = a^{-1} \) is also a real pole/zero.

- Same fact holds for a complex pole/zero.
The problem of finding the transfer function of a filter is the problem of **universal function approximation.** This is usually solved using **basis functions** (Fourier, Chebyshev, ...). In this course, the basis functions are polynomials or rational functions in $z$ (or $z^{-1}$).

**Finite Impulse Response (FIR) filter** — digital filters characterized by transfer functions in the form of a polynomial

$$H(z) = a_0 + a_1 z^{-1} + \ldots + a_M z^{-M}$$
Digital Filter Design

- **Infinite Impulse Response (IIR) filter** — digital filter characterized by transfer functions of the form

\[
H(z) = \sum_{i=0}^{M} a_i z^{-i} \div \sum_{j=0}^{N} b_j z^{-j} = \frac{A(z^{-1})}{B(z^{-1})}
\]
Digital Filter Design

- **FIR filters** are stable and causal.
- **IIR filters** are stable if all poles of $H(z)$ are within the unit circle in the complex plane.
- **IIR filters** are causal if $b_L$ is the first non-zero coefficient in the denominator, i.e. $b_0 = b_1 = \ldots b_{L-1} = 0$
- **Ideal Case:** design of filters with linear phase in the passband.
- **Related problem:** What is the phase in the stopband?
Digital Filters — Magnitude and Phase Characteristics

- Low-pass Filter
- Band-reject Filter
- Band-pass Filter
- All-pass Filter
- High-pass Filter
- Phase Characteristics

(Charts showing magnitude and phase characteristics of different types of digital filters with frequency axes from $-2\pi$ to $2\pi$ and phase characteristics from $-2\pi$ to $3\pi$)
All-pass Digital Filters

- An all-pass filter is an IIR filter with a constant magnitude function for all digital frequency values.
- A transfer function $H(z)$ represents an all-pass filter if for every pole $p_k = r_k e^{j\theta}$ there is a corresponding zero $z_k = \frac{1}{r_k} e^{-j\theta}$.
- The poles and zeros will occur in conjugate pairs if $\theta_k \neq 0$ or $\pi$.
- Let each $H_i(z)$, $1 \leq i \leq P$, be an all-pass filter. Then
  $$H(z) = \prod_{i=1}^{P} H_i(z)$$
  is also an all-pass filter.
All-pass Digital Filters

- All-pass filters are **phase selective** (as opposed to **frequency selective**) and are extremely useful in DSP system design.

- A typical first-order section of an all-pass digital filter has transfer function

\[
H_1(z) = \frac{z^{-1} - a}{1 - az^{-1}}
\]

where \(a\) is real and for stability \(|a| < 1\).

\[
|H_1(e^{j\theta})|^2 = \left| \frac{e^{-j\theta} - a}{1 - ae^{-j\theta}} \right|^2 = 1
\]
All-pass Digital Filters

A typical first-order section of an all-pass digital filter has transfer function

\[
H_2(z) = \frac{1 - \left( \frac{2}{r_k} \right) \cos \theta_k z^{-1} + \left( \frac{1}{r_k} \right) z^{-2}}{1 - 2 r_k \cos \theta_k z^{-1} + r_k^2 z^{-2}}
\]

\[
= \frac{\left[ 1 - \left( \frac{1}{r_k} \right) z^{-1} e^{j \theta_k} \right] \left[ 1 - \left( \frac{1}{r_k} \right) z^{-1} e^{-j \theta_k} \right]}{\left[ 1 - r_k z^{-1} e^{j \theta_k} \right] \left[ 1 - r_k z^{-1} e^{-j \theta_k} \right]}
\]
All-pass Digital Filters

- Poles at
  \[ p_{1,2} = r_\kappa e^{\pm j\theta_\kappa} \]

- Zeros at
  \[ z_{1,2} = \frac{1}{r_\kappa} e^{\pm j\theta_\kappa} \]

- Filter is stable when \(|r_\kappa| < 1\).

- **Exercise:** Show that
  \[ |H_2(e^{j\theta})|^2 = r_\kappa^{-4} = c \]
  where \(c\) is a constant and hence it is an all-pass filter.
FIR Filter Design

- Transfer function is of the form
  \[ H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} \]
  where the impulse response is of length \( N \).
- The filter will have **linear phase** if
  \[ h(n) = h(N - 1 - n) \]
  (symmetry of the impulse response).
If $N$ is odd, it can be shown that

\[
H(e^{j\theta}) = \sum_{n=0}^{N-1} h(n)e^{-jn\theta}
\]

\[
= e^{-j\frac{(N-1)}{2}\theta} \left\{ h\left(\frac{N-1}{2}\right) \right. \\
+ \sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cos \left[\left(n - \frac{N-1}{2}\right)\theta\right] \right\}
\]
FIR Filter Design

- If \( N \) is even, it can be shown that

\[
H(e^{j\theta}) = e^{-j\frac{(N-1)}{2}\theta} \left\{ \sum_{n=0}^{\frac{N-1}{2}} 2h(n) \cos \left[ (n - \frac{N-1}{2})\theta \right] \right\}
\]

- In both cases, the phase \( \phi(\theta) \) of the FIR filter is given by

\[
\phi(\theta) = \frac{N-1}{2}\theta
\]

which is linear

- The group delay is

\[
\tau(\theta) = \phi'(\theta) = \frac{N-1}{2}
\]

which is constant.
FIR Filter Design

- The zero locations of FIR filters are restricted by certain symmetry representations by the symmetric impulse property.
- Write
  \[ H(z) = z^{-(N-1)} \sum_{n=0}^{N-1} h(n)z^{N-n-1} \]
- Let \( m = N - n - 1 \) be a new (dummy) variable. Then
  \[
  H(z) = z^{-(N-1)} \sum_{n=0}^{N-1} h(N - m - 1)z^m \\
  = z^{-(N-1)} \sum_{n=0}^{N-1} h(m)(z^{-1})^{-m} = z^{-(N-1)}H(z^{-1})
  \]
From above, the zeros of $H(z)$ are the zeros of $H(z^{-1})$ except, possibly, for zeros at the origin.

If $z_i = a$ is a real zero of $H(z)$ then $z_i^{-1} = a^{-1}$ is also a zero of $H(z)$.

If $z_i = e^{j\theta_i}$ is a zero of $H(z)$, where $\theta_i \neq 0$ and $\theta_i \neq \pi$, then $z_i^{-1} = \bar{z}_i = e^{-j\theta_i}$ is also a zero of $H(z)$.

If $z_i = r_i e^{j\theta_i}$ is a zero of $H(z)$, where $\theta_i \neq 0$ and $\theta_i \neq \pi$, then $\bar{z}_i = r_i e^{-j\theta_i}$, is also a zero of $H(z)$ as are $z_i^{-1} = \frac{1}{r_i} e^{j\theta_i}$ and $\bar{z}_i^{-1} = \frac{1}{r_i} e^{j\theta_i}$.
FIR Filter Design

An FIR filter can also be written in terms of the Discrete Fourier Transform (DFT)

\[ \tilde{H}(k) = \sum_{n=0}^{N-1} h(n) e^{-j \frac{2\pi nk}{N}} \]

\( \tilde{H}(k) \) is a **uniformly spaced** \( N \) point sample sequence of the digital filter frequency response. Hence the impulse response sequence \( h(n) \) and the transfer function \( H(z) \) are given by

\[ h(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{H}(k) e^{j \frac{2\pi nk}{N}} \]
and

\[ H(z) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{H}(k) \frac{1 - z^{-N}}{1 - z^{-1} e^{j \frac{2\pi k}{N}}} \]

This last expression is the key to the design of an FIR filter.
An FIR Design Example

- Design a low-pass digital filter with the magnitude characteristics shown below. Find an appropriate transfer-function using a 16-point sampling method.
An FIR Design Example

In this case

\[ \tilde{H}(0) = \tilde{H}(1) = \tilde{H}(15) = 1 \]

and

\[ \tilde{H}(k) = 0, \quad k = 2, 4, \ldots, \tilde{H}(14) = 0 \]

Hence

\[ H(z) = \frac{1}{16} \left[ \sum_{k=0}^{15} \frac{(1 - z^{-16})\tilde{H}(k)}{1 - z^{-1}e^{jk\pi/8}} \right] \]
An FIR Design Example

Hence

\[ H(z) = \frac{1 - z^{-16}}{16} \left[ \frac{1}{1 - z^{-1}} + \frac{2(1 - z^{-1} \cos (\frac{\pi}{8}))}{1 - 2z^{-1}\cos (\frac{\pi}{8}) + z^{-2}} \right] \]

The frequency response can be shown that the frequency response of this filter will be equal to the specifications at the sampling frequencies \( \theta = \frac{k\pi}{8}, \ k = 0, 1, \ldots, 15 \).
FIR Design — Windowing Method

- The Fourier series expansion of $H(e^{j\theta})$ is

$$H(e^{j\theta}) = \sum_{n=-\infty}^{\infty} h(n)e^{-jn\theta}$$

$$h(n) = \frac{1}{2\pi} \int_{n=-\pi}^{\pi} H(e^{j\theta})e^{jn\theta} \, d\theta$$

- The infinite series in the penultimate equation can be truncated to obtain the digital filter. However, the Gibbs phenomenon states that the truncation will cause overshoots and ripples in the desired frequency response.
In the windowing method, a finite weighting sequence \( w(n) \), called a windows, is used to obtain the finite impulse response \( h_D(n) \) where

\[
\begin{align*}
h_D(n) &= h(n)w(n)
\end{align*}
\]

where

\[
\begin{align*}
w(n) &= \begin{cases} 
0, & n > N \\
0, & n < 0
\end{cases}
\end{align*}
\]

This design now proceeds as follows.
FIR Design — Windowing Method

- a) Specify the desired frequency response $H(e^{j\theta})$, e.g., by the frequency sampling method.
- b) Construct $h(n)$ as above or by the inverse $z$-transform of $H(z)$ (replacing $e^{j\theta}$ by $z$).
- Use an appropriate $w(n)$ to modify $h(n)$ to obtain the FIR filter’s impulse response sequence.
FIR Design — Windowing Method

- The windowing method smooths out ripples and overshoots in the original response. See below for the simple window

\[
w(n) = \begin{cases} 
1 + \cos \frac{2n\pi}{N}, & 0 \leq n \leq N - 1 \\
0, & \text{otherwise}
\end{cases}
\]