Background

Difference equations express the recursion relationships between the present output sample of the CT signal and

- the present input sample
- past input samples
- past output samples
Background

\[ x(n-1) + x(n-2) + \cdots + x(n-M) \]

\[ x(n) \times x(n-1) \times x(n-2) \times \cdots \times x(n-N) \]

\[ b_0 + b_1 + \cdots + b_M \]

\[ a_1 + a_2 + \cdots + a_N \]

adder

multipliers

unit delay

\[ x(n) \]

\[ b_0 \]

\[ y(n) \]

\[ x(n) \]

\[ T \]

\[ y(n-1) \]

\[ x(n-1) \]

\[ b_1 \]

\[ y(n-2) \]

\[ x(n-2) \]

\[ b_2 \]

\[ y(n-N) \]

\[ x(n-M) \]

\[ b_M \]
The general form here is

\[ y(n) = b_0 x(n) + \sum_{k=1}^{M} b_k x(n - k) - \sum_{j=1}^{N} a_j y(n - j) \]  \hspace{1cm} (1)

or

\[ y(n) + \sum_{j=1}^{N} a_j y(n - j) = \sum_{k=0}^{M} b_k x(n - k) \]  \hspace{1cm} (2)
The presence of past values in the computation of $y(n)$ means that this is a recursive operation. If the past values are not present then it is termed non-recursive. These equations are equivalent to differential equations for CT systems in that they describe the time domain behavior of DT systems. Strictly they should be called recursive equations and are, in fact, easily derived from the true difference equations which are the exact equivalents of differential equations. However, the terminology ‘difference equations’ is now well established and hence its use here.
Background

The most common definition of the **order** of a DT system is the largest value of the **index** \( j \) of the **past output samples** in the system difference equation, i.e. \( N \) in (1) and (2). Alternatively it is the **delay index** of the oldest output sample.

The **initial conditions** are the contents of the output delays at \( n = 0 \), i.e. the set of \( N \) values \( \{ y(-j) \} \) **which comprise the elements of the initial state vector.** It is assumed that \( \{ x(n) \} \) always causal so it has zero initial conditions.

As in the CT case, the solution of (1) or (2) can be written as the **sum of the zero state and zero input responses.**
Background

These are much easier to define than the corresponding singularity functions for CT signals.

Unit Step Sequence

\[ x_s(n) = \begin{cases} 
0, & n < 0 \\
1, & n \geq 0 
\end{cases} \]

Unit Impulse Sequence

\[ x_i(n) = \begin{cases} 
1, & n = 0 \\
0, & n \neq 0 
\end{cases} \]
Background

This last sequence is used to test DT systems in the same way as the unit impulse function in CT systems but is fully realizable. It is also known as the **Kronecker delta function**. It can also be scaled and shifted, e.g.

\[ x(n) = A\delta(n - 3) = \begin{cases} 
  A, & n = 3 \\
  0, & n \neq 0 
\end{cases} \]
Background

These two singularity sequences are related in a similar way as their CT counterparts. In particular, replacing integration by summation, a running summation of the unit impulse sequence yields the unit step sequence, i.e.

\[ x_s(n) = \sum_{k=-\infty}^{n} \delta(n) \]

The **step response** of a DT system is defined as its zero state response to a unit step function and denoted here by \( g(n) \).
Background

The **impulse response** of a DT system is defined as its zero state response to a unit impulse, is denoted by $h(n)$, and is related to the step response by the running summation

$$g(n) = \sum_{k=-\infty}^{n} h(k)$$

It can often be found by inspection.
Examples

Determine the impulse responses for the systems shown in the next figure.

For (a), we have by inspection

\[ h(n) = 0.25 [\delta(n) + \delta(n - 1) + \delta(n - 2) + \delta(n - 3)] \]
\[ = 0.25 [x_s(n) - x_s(n - 4)] \]

(b)
Here we have

\[ h(n) = \begin{cases} 
0, & n < 0 \\
0.25, & n = 0 \\
0.25 \times 0.75, & n = 1 \\
0.25 \times 0.75^2, & n = 2 \\
\vdots
\end{cases} \]
Examples

This can also be expressed as the exponentially decaying sequence

\[ h(n) = 0.25x_s(n)(0.75)^n \]
Examples

The resulting impulse responses

(a)

(b)
Convolution Sum

This is the equivalent of the convolution integral for CT systems and enables us to evaluate the zero state response of a DT system to any input and the mathematical concepts involved are much simpler!

The input sequence can be expressed as the sum of scaled and time shifted unit impulse functions as

\[ x(n) = \sum_k x(k) \delta(n - k) \]

and by superposition the output sequence of a linear system is the sum of similarly scaled and shifted impulse responses, i.e.

\[ y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n - k) \]
Convolution Sum

This is the **basic form of the convolution sum**. As for the convolution integral, the limits on the summation can be changed for the commonly encountered case when both the input sequence \( \{x(n)\} \) and the impulse response \( h(n) \) are **causal**. In which case,

\[
y(n) = \sum_{k=0}^{n} x(k)h(n - k)
\]

if

\[
x(n) = 0, \quad n < 0
\]
\[
h(n) = 0, \quad n < 0
\]

As for CT systems, the convolution sum is often denoted by

\[
y(n) = x(n) \ast h(n).
\]
Convolution Sum

Also a simple change of index gives the following alternative form which is sometimes more convenient to work with.

\[ y(n) = \sum_{k=0}^{n} x(n - k)h(k) \]

As an example, we use the convolution sum to find the step response of the second example given above where setting \( x(n) = x_s(n) \) in the basic form of the convolution sum gives the step response as

\[ g(n) = \sum_{k=-\infty}^{\infty} x_s(k)h(n - k) \]
Convolution Sum

Substituting for the impulse response now gives

\[ g(n) = \sum_{k=-\infty}^{\infty} x_s(k)x_s(n - k)(0.75)^{n-k} \]

Also \( x_s(k) = 0, \ k < 0 \), and \( x_s(n - k) = 0, \ n - k < 0 \), i.e. for \( k > n \) this last expression can be simplified by changing the summation limits.
Convolution Sum

This gives

\[ g(n) = 0.25 \sum_{k=0}^{n} (0.75)^{n-k} \]

\[ = 0.25 \sum_{j=0}^{n} (0.75)^{j}, \ j = n - k \]

\[ = 0.25 \left[ \frac{1 - (0.75)^{n+1}}{1 - 0.75} \right] \]

\[ = 1 - (0.75)^{n+1} \]

\[ \rightarrow 1, \text{ as } n \rightarrow +\infty \] (3)

as before.
z-Transforms

z-transforms for DT signals are equivalent to the Laplace transform for CT signals. In the usual unilateral form it is defined for a sequence \( \{x(n)\} \) by

\[
X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = x(0) + z^{-1}x(1) + z^{-2}x(2) + \cdots
\]

where the ‘frequency variable’ \( z \) is, in general, complex. In the above form there is no apparent connection with the frequency domain and \( z \) seems to act as a time position operator. There is, however, a relationship with the Laplace transform (i.e. the variable \( s \)) which is detailed later in this section.

**Note:** The unilateral (or one-sided) definition implies that \( \{x(n)\} \) is causal.
z-transform convergence

The z-transform defines for each sequence a continuous complex-valued surface over the complex plane \( \mathbb{C} \). For finite sequences, its value is always defined across the entire complex plane. For infinite sequences, it can be shown that the z-transform converges only in the region

\[
\lim_{n \to \infty} \left| \frac{x(n+1)}{x(n)} \right| < |z| < \lim_{n \to -\infty} \left| \frac{x(n+1)}{x(n)} \right|
\]
The $z$-transform identifies a sequence unambiguously only in conjunction with a given region of convergence. In other words, there exist different sequences, that have the same expression as their $z$-transform, but that converge for different amplitudes of $z$. The $z$-transform is a generalization of the Fourier transform, which it contains on the complex unit circle ($|z| = 1$):

$$\mathcal{F}\{x(n)\}(\omega) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$
z-Transforms

Formally the inverse transform is given by a contour integral in the complex $z$-plane but in many cases, including all those considered here, inversion can be achieved (if required) by a table of transform pairs.

**Notation**

\[
\begin{align*}
\text{Transform } Z\{x(n)\} &= X(z) \\
\text{Inverse } Z^{-1}\{X(z)\} &= x(n) \\
\text{Pair } \{x(n)\} &\leftrightarrow X(z)
\end{align*}
\]
Suppose that the DT sequence \( \{x(n)\} \) represents the samples of a causal CT waveform \( x(t) \) at \( t = nT, \ n \geq 0 \). Then, for instantaneous sampling, the resulting sampled signal is given by

\[
x_S(t) = \sum_{n=0}^{\infty} x(nT) \delta(t - nT)
\]

Applying the Laplace transform to this last expression and using the time shift theorem we have that

\[
X_S(s) = \sum_{n=0}^{\infty} x(nT)e^{-snT} = \sum_{n=0}^{\infty} x(n)z^{-n}
\]

where \( z = e^{sT} \).
Relationship with the Laplace Transform

Hence we have the following general relationship between the ZT of the sequence \( \{x(n)\} \) and the LT of the underlying sampled CT signal, i.e.

\[
X(z) = X_S(s)\big|_{z=e^{sT}}
\]

Now set \( s = \sigma + i\omega \) and hence

\[
z = e^{(\sigma+i\omega)T} = e^{\sigma T}e^{i\omega T}
\]

i.e. \( |z| = e^{\sigma T} \) and \( \angle z = \omega T \). Setting \( \sigma = 0 \) in the first of these expressions now shows that the \( i\omega \) axis in the \( s \)-plane maps to the unit circle in the \( z \)-plane and that the open left half of the \( s \)-plane (\( \sigma < 0 \)) maps into the interior of the unit circle in the \( z \)-plane.
Relationship with the Laplace Transform

\[ \text{LHP} \]

- jω
- s-plane
- \( \sigma \)

\[ \text{unit circle} \]

- Im
- z-plane
- Re
- \( 1 \)
- \( \omega T \)
- \( \text{unit circle} \)
Evaluating the $z$ Transform

First we give some simple (but relevant) cases which can be evaluated directly from the defining summation.

**Unit step sequence**

$$Z\{x_s(n)\} = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z - 1}, \ |z| > 1$$

**Unit impulse**

$$Z\{\delta(n)\} = z^0 = 1$$
Evaluating the $z$ Transform

Delayed unit step sequence

$$Z\{\delta(n - k)\} = z^{-k}$$

$$Z\{a^n\} = \frac{z}{z - a}, \quad |z| > a$$

Note that there are restrictions on the magnitude of $z$ in some cases to ensure that the defining sum actually converges. The delayed unit step case also shows that $z$ acts as a time shift operator - note that $z^{-k} = e^{-skT} \Rightarrow \text{delay} = kT$. 

Professor Eric Rogers
ELEC 6218 — Signal Processing — $z$ Transform Analysis
Evaluating the $z$ Transform

In practice, use of the $z$ transform follows that for the Fourier and Laplace transforms in that a table of transform pairs is used. Note that to use this table to find inverse transforms, the functions in the left-hand column should be regarded as causal and therefore should be multiplied by the unit step function.
### z Transform Table

<table>
<thead>
<tr>
<th>$x(n)$</th>
<th>$X(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(n)$</td>
<td>1</td>
</tr>
<tr>
<td>$x_s(n)$</td>
<td>$\frac{z}{z-1}$</td>
</tr>
<tr>
<td>$n$</td>
<td>$\frac{z}{(z-1)^2}$</td>
</tr>
<tr>
<td>$n^2$</td>
<td>$\frac{z(z+1)}{(z-1)^3}$</td>
</tr>
<tr>
<td>$a^n$</td>
<td>$\frac{z}{z-a}$</td>
</tr>
<tr>
<td>$n a^n$</td>
<td>$\frac{z a}{(z-a)^2}$</td>
</tr>
<tr>
<td>$\sin(\Omega n)$</td>
<td>$\frac{z \sin \Omega}{z^2 - 2z \cos \Omega + 1}$</td>
</tr>
<tr>
<td>$\cos(\Omega n)$</td>
<td>$\frac{z^2 - z \cos \Omega}{z^2 - 2z \cos \Omega + 1}$</td>
</tr>
<tr>
<td>$a^n \cos(\Omega n)$</td>
<td>$\frac{a z \sin \Omega}{z^2 - 2a z \cos \Omega + a^2}$</td>
</tr>
<tr>
<td>$a^n \cos(\Omega n)$</td>
<td>$\frac{z^2 - a z \cos \Omega}{z^2 - 2a z \cos \Omega + a^2}$</td>
</tr>
</tbody>
</table>
$z$ Transform Properties

Property 1 — Linearity

\[ Z\{x(n) + y(n)\} = X(z) + Y(z) \]
\[ Z\{k x(n)\} = kX(z) \]

Property 2 — Time shift (delay)

\[ Z\{x(n - m)\} = z^{-m}X(z) + \sum_{k=1}^{m} x(-k)z^{k-m} \]
\[ x_s(n - m)x(n - m) = z^{-m}X(z) \]

Property 3 — Frequency shift

\[ Z\{a^n x(n)\} = X\left(\frac{z}{a}\right) \]
**$z$ Transform Properties**

**Property 4 — Time reversal**

$$Z\{x(-n)\} = X(z^{-1})$$

**Property 5 — Summation**

$$Z\left\{\sum_{k=0}^{n} x(k)\right\} = \frac{z}{z-1} X(z)$$

**Property 6 — Convolution (provided both sequences involved are causal)**

$$Z\{x(n) \ast y(n)\} = X(z)Y(z)$$

**Property 7 — Initial Value**

$$Z\{x(0)\} = \lim_{z \to \infty} [X(z)]$$
Property 8 — Final value

\[ \lim_{n \to +\infty} \{x(n)\} = \lim_{z \to 1} [(1 - z^{-1})X(z)] \]

provided all poles of the expression on the right hand side lie outside the unit circle.
z Transform Properties

- In the case of Property 2, it is only necessary to remember the second form since most sequences are assumed to be causal.
- Property 3 — setting $a = e^{\beta t}$ gives $\frac{z}{a} = \frac{e^{sT}}{e^{\beta T}} = e^{(s-\beta)T}$.
- Property 5 is equivalent to the Laplace transform integration theorem for CT signals.
- Property 6 - the convolution theorem — is essential for system response analysis.
Solving Difference Equations

Taking the ZT of both sides of a difference equation gives an algebraic expression in $z$ (or $z^{-1}$) which can then be manipulated to yield the ZT of the response. This follows immediately on using the ZT of the input sequence and Properties 1 and 2. There are two cases to consider.

**Case 1: non-zero initial conditions - must use the first form of Property 2.**

**Example:** Use the ZT to solve the difference equation

$$y(n) - 1.4y(n - 1) + 0.48y(n - 2) = x(n)$$

for $x(n) = x_s(n)$ and initial conditions $y(-1) = 5, \ y(-2) = 0$. 
Solving Difference Equations

Taking the ZT and using Property 2 we have that

\[ Y(z) - 1.4 [z^{-1}Y(z) + y(-1)] + 0.48 [z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = X(z) \]

Now consider the case when

\[ X(z) = \frac{z}{z - 1} \]

and hence

\[ Y(z) = \frac{z^3}{(z - 1)(z^2 - 1.4z + 0.48)} + \frac{z(7z - 2.4)}{(z^2 - 1.4z + 0.48)} \]
Solving Difference Equations

By partial fractions we now have that

\[ Y(z) = \frac{12.5z}{z - 1} - \frac{4z}{z - 0.8} + \frac{1.5z}{z - 0.6} \]

and using Table 2

\[ y(n) = x_s(n)(12.5 - 4 \times (0.8)^n + 1.5 \times (0.6)^n) \]
Solving Difference Equations

Now use Properties 7 and 8 for this example. Here

\[ y(0) = 12.5 - 4 + 1.5 = 10 \]

and by Property 7

\[ y(0) = \lim_{z \to \infty} [Y(z)] = 10 \]

Also

\[ y(\infty) = 12.5 \]

and by Property 8

\[ y(\infty) = \lim_{z \to \infty} [(z - 1)Y(z)] = \lim_{z \to 1} \left[ \frac{z^3}{(z - 0.8)(z - 0.6)} \right] = 12.5 \]
Solving Difference Equations

Zero initial conditions.
We have already seen that the convolution sum gives the system response in this case, i.e.

\[ y(n) = x(n) \ast h(n) \]

and if (as assumed here) \( \{x(n)\} \) and \( h(n) \) are causal we have on applying the ZT to this last expression gives

\[ Y(z) = G(z)X(z) \]

where

\[ G(z) = \frac{Y(z)}{X(z)} = Z\{h(n)\} \]

is the transfer function of the DT system.
Solving Difference Equations

The transfer function of a DT system can be computed by any one of the following methods.

- taking the ZT of the impulse response — which, in turn, can often be found by inspection.
- applying the ZT to both sides of the defining difference equation.
- by block diagram algebra applied to the block diagram of the system.
Solving Difference Equations

Determine the transfer function of the system shown in the figure below by the second method given above. Assume that the input sequence \( \{x(n)\} \) is causal and \( 0 < |\beta| < 1 \).

The system difference equation is

\[
y(n) = x(n) - x(n - 1) + \beta y(n - 1)
\]

Rearranging and applying the ZT gives

\[
Y(z) - \beta z^{-1} Y(z) = X(z) - z^{-1} X(z)
\]

Hence

\[
G(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \beta z^{-1}} = \frac{z - 1}{z - \beta}
\]
Solving Difference Equations

Suppose that we write

\[ G(z) = \frac{b(z)}{a(z)} \]

Then the poles and zeros of \( G(z) \) are defined as in the CT case, i.e. the **system poles** are the roots of

\[ a(z) = 0 \]

and the **system zeros** as the roots of

\[ b(z) = 0 \]
Solving Difference Equations

Also it follows from the analysis above that the system is stable if, and only if, all poles have modulus strictly less than unity. Hence the system here is stable since we have assumed that $|\beta| < 1$. 
Summary of the Transform Domain

- Fourier transform:

  \( x(t) \rightarrow \bullet X(j\omega) \)

  \[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \]  
  \[ x[n] \rightarrow \bullet X(e^{j\Omega}) \]

  \[ X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \] (5)

  With angular frequency \( \omega = 2\pi f \) and normalised angular frequency \( \Omega = \frac{2\pi \omega}{\omega_S} \) for sampling at \( \omega_S \).

- Note that \( X(e^{j\Omega}) \) is periodic with \( 2\pi \).

- Z-transform: \( H(z) \rightarrow \circ h[n] \) such that \( H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \).