ELEC6229
Advanced Systems and Signal Processing
Kalman Filtering

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By the end of this week’s lectures, you should be able to

- Use Kalman filtering methods to estimate system states

Reading: Sections 4.1, 5.1-5.3, 13.2.3 (Dan’s book)
Lecture Overview

- Motivation
- State estimation problem
- Kalman filtering
- Extended Kalman filtering
System State Estimation

State estimation: to estimate the states $x(t)$

- Critical in system analysis and design
- Kalman filtering and Particle filtering
An Example

- How could we ‘reveal’ the state information from the noisy measurements?
Kalman Filter

Rudolf E. Kálmán
State Estimation Problem

Estimate past state (smoothing)

\[ x(t_0) \rightarrow x(t) \]

Estimate current state (filtering)

Estimate future state (prediction)

\[ y(t_0) \rightarrow y(t) \]
Filtering: A General Formulation

- The model

\[ x_{k+1} = f_k(x_k, u_k, w_k) \]
\[ y_k = h_k(x_k, u_k, v_k) \]

where \( w_k \sim N(0, Q_k) \) and \( v_k \sim N(0, R_k) \) are uncorrelated white noise sequences.

- The problem: estimate the current state \( x_k \) based on the current and past input and observed outputs \( u_k, y_k, k = 0, 1, \ldots, k \).
The model

\[ x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1} \]

\[ y_k = H_kx_k + v_k \]

where \( w_k \sim N(0, Q_k) \) and \( v_k \sim N(0, R_k) \) are uncorrelated white noise sequences.

The problem: estimate the current state \( x_k \) based on the current and past input and observed outputs \( u_k, y_k, k = 0, 1, \ldots, k \).
Optimal State Estimation Problem

- Obtain an optimal estimate of $x_k$ to minimise

$$J_k = E[(\hat{x}_k - x_k)^T (\hat{x}_k - x_k)]$$

- Rudolf R. Kalman solved in this problem around 1960

- In his seminal work:
  - If the noises are zero mean uncorrelated Gaussian white noises, Kalman filter is the best among all linear and nonlinear filters.
  - Kalman filter is the optimal linear filter.
Optimal State Estimation Problem

- Obtain an optimal estimate of $x_k$ to minimise
  $$J_k = E[(\hat{x}_k - x_k)^T(\hat{x}_k - x_k)]$$

- At time $k - 1$, we have
  - The estimation $\hat{x}_{k-1}$
  - Estimation error covariance $P_{k-1}$

- At time $k$, we have
  - New measurement $y_k = H_k x_k + v_k$
  - Task: to obtain $\hat{x}_k$ and $P_k$
Recall Recursive Least Square

- Obtain an optimal estimate of $x_k$ to minimise
  \[ J_k = E[(\hat{x}_k - x_k)^T(\hat{x}_k - x_k)] \]

- At time $k - 1$, we have
  - The estimation $\hat{x}_{k-1}$
  - Estimation error covariance $P_{k-1}$

- At time $k$, we have
  - New measurement $y_k = H_k x + v_k$
  - Obtain the estimate as
    \[ \hat{x}_k = \hat{x}_{k-1} + K_k(y_k - H_k \hat{x}_{k-1}) \]
Comparing RLS with KF

- Obtain an optimal estimate of $x_k$ such that
  \[ J_k = E[(\hat{x}_k - x_k)^T(\hat{x}_k - x_k)] \]

- Measurement: $y_k = H_k x + v_k$

- RLS: estimate a constant parameter $x$ (Note: $\hat{x}_{k-1}, P_{k-1}$ are about $x$)

- Optimal State Estimation: estimate a state $x_k$
  \[ x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1} \]
  Note: $\hat{x}_{k-1}, P_{k-1}$ are about $x_{k-1}$
Kalman Filtering: Prediction

- Optimal State Estimation: estimate a state $x_k$
  
  $$x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1}$$
  
  Note: $\hat{x}_{k-1}, P_{k-1}$ are about $x_{k-1}$

- Prediction: obtain a priori estimate $\hat{x}_k^-$ of $x_k$ using $\hat{x}_{k-1}, P_{k-1}$ as
  
  $$\hat{x}_k^- = F_{k-1}\hat{x}_{k-1} + G_{k-1}u_{k-1}$$
  $$P_k^- = F_{k-1}P_{k-1}F_{k-1}^T + Q_{k-1}$$
  
  Note: $\hat{x}_k^-, P_k^-$ are about $x_k$ before $y_k$ arrives
Kalman Filtering: Correction

- Correction: obtain a posterior estimate $\hat{x}_k$ of $x_k$ using $\hat{x}_k^-, P_k^-$ as

$$\hat{x}_k = \hat{x}_k^- + K_k (y_k - H_k \hat{x}_k^-)$$

$$P_k = (I - K_k H_k) P_{k-1} (I - K_k H_k)^T + K_k R_k K_k^T$$

where

$$K_k = P_{k-1} H_k^T (H_k P_{k-1} H_k^T + R_k)^{-1}$$

- $\hat{x}_k^-, P_k^-$ are about $x_k$ after $y_k$ arrives: $\hat{x}_k^+, P_k^+$

- Note: the derivation process is the same as RLS.
Kalman Filtering Algorithm

- At time $k - 1$, we have estimate $\hat{x}_{k-1}^+$ and $P_{k-1}^+$
- At time $k$, new measurement $y_k$ arrives, obtain the estimate of $x_k$ as follows:

**Stage 1: prediction (a prior estimate)**

$$\hat{x}_k^- = F_{k-1} \hat{x}_{k-1}^+ + G_{k-1} u_{k-1}$$

$$P_k^- = F_{k-1} P_{k-1}^+ F_{k-1}^T + Q_{k-1}$$

**Stage 2: correction (a posterior estimate)**

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1}$$

$$\hat{x}_k^+ = \hat{x}_k^- + K_k (y_k - H_k \hat{x}_k^-)$$

$$P_k^+ = (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T$$
Kalman Filtering Algorithm

- Initial choice of estimate $\hat{x}_0$ and $P_0$
  - $\hat{x}_0$ ideally should be good enough
  - $P_0$ should reflect the confidence in $\hat{x}_0$ and should be positive definite.

- Other properties:
  - Will $P_k^+$ converge?
  - Steady state Kalman filter.
The model:

\[ x_{k+1} = \begin{bmatrix} 1 & 0.025 \\ 0 & 1 \end{bmatrix} x_k + w_k, \quad y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k \]
A nonlinear model

\[ x_{k+1} = f_k(x_k, u_k, w_k) \]

\[ y_k = h_k(x_k, u_k, v_k) \]

where \( w_k \sim N(0, Q_k) \) and \( v_k \sim N(0, R_k) \) are uncorrelated white noise sequences.

The problem: estimate the current state \( x_k \) based on the current and past input and observed outputs \( u_k, y_k, k = 0, 1, \ldots, k \).
Extended Kalman Filtering

- At time $k-1$, we have estimate $\hat{x}^+_{k-1}$ and $P^+_{k-1}$
- At time $k$, new measurement $y_k$ arrives, obtain the estimate of $x_k$ as follows (via linearisation):

  **Stage 1: prediction (a prior estimate)**

  $$\hat{x}^-_k = f_{k-1}(x^+_{k-1}, u_{k-1}, 0)$$
  $$P^-_k = F_{k-1} P^+_{k-1} F^T_{k-1} + L_{k-1} Q_{k-1} L^T_{k-1}$$

  where

  $$F_{k-1} = \frac{\partial f_{k-1}}{\partial x} \bigg| \hat{x}^+_{k-1}, L_{k-1} = \frac{\partial f_{k-1}}{\partial w} \bigg| \hat{x}^+_{k-1}$$
Stage 2: correction (a posterior estimate)

\[ \hat{x}_k^+ = \hat{x}_k^- + K_k(y_k - h_k(\hat{x}_k^-, u_k, 0)) \]

(Question: How to derive \( K_k \))

\[ K_k = P_k^- H_k^T (H_k P_k^- H_k^T + M_k R_k M_k^T)^{-1} \]

\[ P_k^+ = (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T \]

where

\[ H_k = \frac{\partial h_k}{\partial x} |_{\hat{x}_k^-}, \quad M_k = \frac{\partial h_k}{\partial u} |_{\hat{x}_k^-} \]
Extended Kalman Filtering

- At time $k - 1$, we have estimate $\hat{x}_{k-1}^+$ and $P_{k-1}^+$
- At time $k$, new measurement $y_k$ arrives, obtain the estimate of $x_k$ as follows (via linearisation):
  
  **Stage 1: prediction (a prior estimate)**
  \[
  \hat{x}_k^- = f_{k-1}(x_{k-1}^+, u_{k-1}, 0) \\
  P_k^- = F_{k-1}P_{k-1}^+F_{k-1}^T + L_{k-1}Q_{k-1}L_{k-1}^T
  \]

  **Stage 2: correction (a posterior estimate)**
  \[
  K_k = P_k^-H_k^T(H_kP_k^-H_k^T + M_kR_kM_k^T)^{-1} \\
  \hat{x}_k^+ = \hat{x}_k^- + K_k(y_k - h_k(\hat{x}_k^-, u_k, 0)) \\
  P_k^+ = (I - K_kH_k)P_k^-(I - K_kH_k)^T + K_kR_kK_k^T
  \]
By the end of this week’s lectures, you should be able to

- Use Kalman filtering methods to estimate system states

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