ELEC6229
Advanced Systems and Signal Processing
Particle Filtering

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By the end of this week’s lectures, you should be able to

- Use particle filtering methods to estimate system states

Reading: Chapter 15 (Dan’s book)
Lecture Overview

- Nonlinear estimation problem
- Bayesian state estimation
- Particle filtering
- Implementation issues
System State Estimation

- State estimation: to estimate the states $x(t)$
  - Critical in system analysis and design
  - Kalman filtering and Particle filtering

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$
Nonlinear State Estimation

- A nonlinear model

\[ x_{k+1} = f_k(x_k, w_k) \]
\[ y_k = h_k(x_k, v_k) \]

where \( w_k \sim N(0, Q_k) \) and \( v_k \sim N(0, R_k) \) are uncorrelated white noise sequences.

- The problem: estimate the current state \( x_k \) based on the current and past input and observed outputs \( y_k, k = 0, 1, \ldots, k \).
Figure 15.1  An example of a multimodal probability density function. What single number should be used as an estimate of $x$?
Bayesian State Estimation

- **Objective**: to recursively estimate the conditional pdf of $x_k$ based on measurements $Y_k: y_1, y_2, \ldots, y_k$

$$p(x_k|Y_k) := p(x_k|y_1, y_2, \ldots, y_k)$$

- The initial estimate is the pdf of $x_0$

$$p(x_0) = p(x_0|Y_0)$$

- Once we know the pdf $p(x_k|Y_k)$:
  - You can obtain the estimate $\hat{x}_k$
  - Expectation, MAP \ldots
Bayesian State Estimation

- At time $t = k - 1$, we have
  
  $$p(x_{k-1}|Y_{k-1}) := p(x_{k-1}|y_1, y_2, \ldots, y_{k-1})$$

- The problem: at time $t = k$, with new measurement $y_k$ to find
  
  $$p(x_k|Y_k) := p(x_k|y_1, y_2, \ldots, y_k)$$

- How could we solve this problem?
  
  - Prediction and update approach
Bayesian State Estimation

- At time $t = k - 1$, we have
  \[ p(x_{k-1}|Y_{k-1}) := p(x_{k-1}|y_1, y_2, ..., y_{k-1}) \]

- **Prediction:**
  \[ p(x_k|Y_{k-1}) \]

- We now have pdf $p(x_k|Y_{k-1})$
Bayesian State Estimation

- We have pdf \( p(x_k|Y_{k-1}) \):

- Update (correction):

\[
p(x_k|Y_k)
\]
Bayesian State Estimation

- Update (continue):

\[ p(y_k|Y_{k-1}) = \int p(y_k|x_k)p(x_k|Y_{k-1}) \, dx_k \]

- Then

\[
p(x_k|Y_k) = \frac{p(y_k|x_k)p(x_k|Y_{k-1})}{p(y_k|Y_{k-1})} = \frac{\int p(y_k|x_k)p(x_k|Y_{k-1}) \, dx_k}{\int p(y_k|x_k)p(x_k|Y_{k-1}) \, dx_k}
\]
Recursive Bayesian State Estimator

- **System model:**
  \[ x_{k+1} = f_k(x_k, w_k), \quad y_k = h_k(x_k, v_k) \]

- **At time** \( k - 1 \), we have \( p(x_{k-1} | Y_{k-1}) \)

- **At time** \( k \), we have new measurement \( y_k \)
  - **Prediction**
    \[ p(x_k | Y_{k-1}) = \int p(x_k | x_{k-1}) \ p(x_{k-1} | Y_{k-1}) \ dx_{k-1} \]
  - **Update**
    \[ p(x_k | Y_k) = \frac{p(y_k | x_k) p(x_k | Y_{k-1})}{\int p(y_k | x_k) p(x_k | Y_{k-1}) \ dx_k} \]
Recursive Bayesian State Estimator

- Once we know $p(x_k|Y_k)$:
  - We can have various estimate $\hat{x}_k$
  - Expectation, MAP ...

- Kalman filter can be derived from this framework:
  - Is it possible?
  - How?

- Difficulties with Bayesian state estimator
Motivation for Particle Filtering

- **System model:**
  \[ x_{k+1} = f_k(x_k, w_k), \quad y_k = h_k(x_k, v_k) \]

- At time \( k - 1 \), we have \( p(x_{k-1}|Y_{k-1}) \)

- At time \( k \), we have new measurement \( y_k \)
  
  - **Prediction**
    \[
    p(x_k|Y_{k-1}) = \int p(x_k|x_{k-1}) p(x_{k-1}|Y_{k-1}) \, dx_{k-1}
    \]

  - **Update**
    \[
    p(x_k|Y_k) = \frac{p(y_k|x_k)p(x_k|Y_{k-1})}{\int p(y_k|x_k)p(x_k|Y_{k-1}) \, dx_k}
    \]
Basic Idea for Particle Filtering

- Particle filter is a technique for implementing recursive Bayesian filter by **Monte Carlo** sampling.
- The idea: represent the posterior density by a set of random particles with associated weights.
The idea: represent the posterior density by a set of random particles with associated weights

\[ p(x) \approx \sum_{i=1}^{N} \frac{1}{N} \delta(x - x^i) \]
Particle Filtering: Importance Sampling

- The idea: represent the posterior density by a set of random particles with associated weights

\[ p(x) \approx \sum_{i=1}^{N} \omega_i \delta(x - x^i) \]
Random Samples and pdf
Particle Filtering: Prediction

- **System model:**
  \[ x_{k+1} = f_k(x_k, w_k), \quad y_k = h_k(x_k, v_k) \]

- **At time** \( k - 1 \), we have \( N \) particles \( \{x_{k-1}^+, 1/N\} \)
  \[ p(x_{k-1} | Y_{k-1}) \approx \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_{k-1,i}^+) \]

- **Prediction:** at time \( k \), generate \( N \) a priori particles
  \[ x_{k,i}^- = f_{k-1}(x_{k-1}^+, w_{k-1}^i) \]
  based on the pdf of \( w_{k-1} \) (the weights do not change)
Particle Filtering: Update

- **Update**: at time $k$, update the weights based on the new measurement $y_k$

  $$q_i = p(y_k | x_{k,i}^-), \quad i = 1, 2, \ldots, N$$

  based on system output equation and pdf of $v_k$.

- **Normalise the weights**

  $$q_i = \frac{q_i}{\sum_i^N q_i}, \quad i = 1, 2, \ldots, N$$

- **The posterior particles are** $\{x_{k,i}^+, q_i\}$
Particle Filtering: Resampling

- The posterior particles are \( \{x_{k,i}^+, q_i\} \)

\[
p(x_k|Y_k) \approx \sum_{i=1}^{N} q_i \delta(x - x_{k,i}^+)
\]

- Calculate your desired statistical measure (N.B. estimate) based on the above pdf

- Resampling: generate new a set of particles

\[
\left\{x_{k,i}^+, \frac{1}{N}\right\}
\]

from \( p(x_k|Y_k) \)
Particle Filtering

\[
\{ x_{k-1,i}^+, 1/N \} \quad \{ x_{k,i}^+, q_i \} \\
\{ x_{k,i}^+, 1/N \} \quad \{ x_{k+1,i}^+, q_i \} \quad \{ x_{k+1,i}, 1/N \}
\]
Particle Filtering: Implementation Issues

- Sample degeneration and impoverishment
  - Resampling schemes
- Roughening
- Regularised particle filtering
- Markov Chain Monte Carlo resampling
- Auxiliary particle filtering
- Particle filtering with other filters
Summary

- Nonlinear estimation problem
- Bayesian state estimation
- Particle filtering
- Implementation issues
By the end of this week’s lectures, you should be able to

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