Introduction to control systems and presentation of the course
Scheme of a closed-loop control system

- Think controller=driver, plant=car;

- Want to maintain certain speed (final output), equal to leftmost input (desired speed), applying a control input (gas pedal force) to plant;

- Even if plant model is perfectly known, it may be different from what is at hand! Think smooth vs. rough road, for example.

- Also, unknown disturbances (e.g. head wind) may affect the system.
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Even if plant model is *perfectly known*, it may be different from what is at hand! Think smooth vs. rough road, for example.

Also, unknown disturbances (e.g. head wind) may affect the system.

This is not a smart way of controlling a system—rather like driving with your eyes closed!
Driving with open eyes;

Uncertainties and disturbances (up to a point) can be accommodated;

Mental sanity does not automatically imply good performance: over- or undershooting is still possible!
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This is a smarter way of controlling a system
The state paradigm and its advantages

• The transfer function describes the input-output relation only under the assumption that the system is initially at rest (i.e. the initial conditions are zero);
• Internal properties also important for input-output behavior are made evident;
• Multivariable systems dealt with in a straightforward way;
• Linear algebra makes simulation, control, and design algorithms simple, intuitive, and easy to implement.
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Areas of application

- Mechanical systems;
- Electrical circuits;
- Power generation and distribution;
- Electrical motors;
- Manufacturing;
- Vehicle engine management;
- ...
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Control theory is victim of its own success: its principles and techniques are so embedded in knowledge and in applications as to be invisible!
The course: aim

Develop the ability to design multivariable control systems and evaluate their performance.

These skills require **mathematics** (theory) but can only put to the test by applying them to concrete problems. The right tool is **Matlab** and its toolboxes.
The course: material

- Lecture slides: downloadable from course Web page. *Work in (eternal) progress.*
- Textbooks (available in library): see course Web page.
- Matlab & toolboxes: on the lab computers.
- Past exam papers: see course Web site.
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The course: you

• Lectures are not a religious rite;
• Knowledge is cumulative;
• The accumulation of knowledge requires action.
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Participate, Revise, Exercise
Coursework, labs and feedback

Coursework (UG):
2 pieces of coursework, due
• Week 7
• Week 11
In total 20% of total marks

Coursework (MSc):
2 pieces of coursework, due
• Week 7
• Week 11
In total 10% of total marks

Directed reading assignment (MSc):
1 piece of coursework, due Week 8, in total 10% of total mark;

Exam: 2 hours, 80% of total marks
Glossary and background reading

- Determinant – 行列式
- Nonsingular/singular matrix – 非奇异/奇异矩阵
- Eigenvector/eigenvalue 特征向量/特征值
- Linear independence/linearly independent 线性无关
- Poles 极点
- Zeroes 零点

Need to master concepts listed in

ELEC3205–6243 Mathematical Background list

on course website
State representations
State: the basic idea

To follow from now on, just look at the chessboard
State: the basic idea

To follow from now on, just look at the chessboard

- The state contains all the **relevant information** about the **future** behavior of the system
- The state is the **memory** of the system
- Independence of past and future **given** the state
Two points of view on state and state representations:

1) The state is \textit{given} as a “natural” set of attributes splitting past and future of the system;
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2) The state is **constructed** from the differential equations/transfer function of the system.
State: two points of view

Two points of view on state and state representations:

1) The state is **given** as a “natural” set of attributes splitting past and future of the system;

2) The state is **constructed** from the differential equations/transfer function of the system.

Contradictory point of views, in some sense, but each with its merits.
“Natural” state: a mechanical example

\[ m_1 \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} + K(x_1 - x_2) = 0 \]

\[ -Kx_1 + m_2 \frac{d^2 x_2}{dt^2} + Kx_2 = F \]
“Natural” state: a mechanical example

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\[
-Kx_1 + m_2 \frac{d^2}{dt^2} x_2 + Kx_2 = F
\]

Can find transfer function between \( F \) and \( x_2 \) via Laplace transform, obtaining transfer function model
Define the state

\[ z := \begin{bmatrix} x_1 \\ \frac{d}{dt} x_1 \\ x_2 \\ \frac{d}{dt} x_2 \end{bmatrix} \]

often called “configuration variables”.

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\[
\frac{d}{dt} z = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{K}{m_1} & 0 & -\frac{D}{m_1} & 0 \\
0 & 0 & 0 & \frac{K}{m_2} \\
0 & 0 & -\frac{K}{m_2} & 0
\end{bmatrix} z + \begin{bmatrix}
0 \\
0 \\
0 \\
\frac{1}{m_2}
\end{bmatrix} F
\]

\[
y = x_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} z
\]
“Natural” state: an electrical example

Choose as state variables $v_C$ and $i_L$, associated with energy storage (or “memory”):

$$x := \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$
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Standard manipulations yield

$$\frac{d}{dt} x = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v$$
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If output is \( i_R \) (note: not a state variable):

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The state equations

General form of (linear, time-invariant) state equations:

\[
\frac{dx}{dt} = Ax + Bu
\]
\[
y = Cx + Du
\]

where \( x : \mathbb{R} \rightarrow \mathbb{R}^n \) is the state, \( y : \mathbb{R} \rightarrow \mathbb{R}^p \) is the output, \( u : \mathbb{R} \rightarrow \mathbb{R}^m \) is the input.
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Note \( A \in \mathbb{R}^{n\times n} \), \( B \in \mathbb{R}^{n\times m} \), \( C \in \mathbb{R}^{p\times n} \), \( D \in \mathbb{R}^{p\times m} \).
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Possible to say more about the function spaces where \( x, \; y \) and \( u \) live...
The state equations

General form of (linear, time-invariant) state equations:

\[
\begin{align*}
\frac{d}{dt} x &= Ax + Bu \\
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State equations describe the system: eliminate \( x \) through algebraic manipulations, obtain a differential equation; Laplace transform it, obtain system transfer function.
Solution of the state equations

Solution of \( \frac{d}{dt} x = Ax + Bu \) with initial condition \( x(0) \):

\[
x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) \, d\tau
\]

where the matrix exponential \( e^{At} \) is defined as

\[
e^{At} := I_n + At + A^2 \frac{t^2}{2} + \cdots + A^k \frac{t^k}{k!} + \cdots
\]
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$$e^{At} := I_n + At + A^2 \frac{t^2}{2} + \cdots + A^k \frac{t^k}{k!} + \cdots$$

The proof is based on:

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \left( I_n + At + A^2 \frac{t^2}{2} + \cdots + A^k \frac{t^k}{k!} + \cdots \right)$$

$$= A + A^2 t + \cdots + A^k \frac{t^{k-1}}{(k-1)!} + \cdots$$

$$= A \left( I_n + At + A^2 \frac{t^2}{2} + \cdots + A^k \frac{t^k}{k!} + \cdots \right) = Ae^{At} = e^{At} A$$
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\]
Solving the state equation efficiently

Calculating matrix exponential via series impossible: need help!
Solving the state equation efficiently-1

Calculating matrix exponential via series impossible: need help!

...as will often happen in the course, Linear Algebra is the white knight!
Solving the state equation efficiently

Recall definition of eigenvalue and eigenvector:

\[ \mathbf{A} \mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda \]

where eigenvalue \( \lambda \in \mathbb{C} \), eigenvector \( \mathbf{v}_\lambda \in \mathbb{C}^n \). The \( \lambda \)s are the roots of the characteristic polynomial:

\[ \chi_A(s) := \det(sI - A) \]
Solving the state equation efficiently-1

If $A$ has all distinct $\lambda$s, then the eigenvectors are linearly independent. Assume this for the moment; it follows

$$A\begin{bmatrix} v_{\lambda_1} & v_{\lambda_2} & \cdots & v_{\lambda_n} \end{bmatrix} = \begin{bmatrix} v_{\lambda_1} & v_{\lambda_2} & \cdots & v_{\lambda_n} \end{bmatrix} =: T$$

and since $T$ is nonsingular,

$$A = T\Delta T^{-1}$$

But then

$$A^k = T\Delta T^{-1} T\Delta T^{-1} \cdots T\Delta T^{-1} = T\Delta^k T^{-1} = T$$
Solving the state equation efficiently-2

It follows that

\[ e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & e^{\lambda_n t} \end{bmatrix} T^{-1} \]
Solving the state equation efficiently-2

It follows that

$$e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix} T^{-1}$$

Knowing the particular analytic form of $u(\cdot)$, the solution of the state equation is a matter of integration:

$$x(t) = T \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} T^{-1} x(0)$$

$$+ T \int_0^t \begin{bmatrix} e^{\lambda_1 (t-\tau)} & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n (t-\tau)} \end{bmatrix} T^{-1} Bu(\tau) d\tau$$
Exercise

Solve

\[ \frac{d}{dt} x = \begin{bmatrix} -2 & 0 \\ 4 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \]

for \( x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( u(\cdot) = 1(\cdot) \), the Heaviside step.
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for \( x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( u(\cdot) = 1(\cdot), \) the Heaviside step.

Matlab commands to use: eig.
Solution

\[
A = \begin{bmatrix}
-1 & 0 \\
4 & 1
\end{bmatrix}
\begin{bmatrix}
-2 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
4 & 1
\end{bmatrix}^{-1}
\]

so that

\[
e^{At} = \begin{bmatrix}
-1 & 0 \\
4 & 1
\end{bmatrix}
\begin{bmatrix}
e^{-2t} & 0 \\
0 & e^{-t}
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
4 & 1
\end{bmatrix}^{-1}
= \begin{bmatrix}
e^{-2t} & 0 \\
-4e^{-2t} + 4e^{-t} & e^{-t}
\end{bmatrix}.
\]
Solution

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A = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix}^{-1}
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\]

Free response is \(e^{At}x(0) = \begin{bmatrix} e^{-2t} \\ -4e^{-2t} + 6e^{-t} \end{bmatrix} \cdot\)
Solution

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Free response is \( e^{At}x(0) = \begin{bmatrix} e^{-2t} \\ -4e^{-2t} + 6e^{-t} \end{bmatrix} \).

Forced response is \( \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \), i.e.

\[
\begin{bmatrix}
\int_0^t e^{-2(t-\tau)} \\
-4e^{-2(t-\tau)} + 4e^{-(t-\tau)}
\end{bmatrix}
\frac{\begin{bmatrix} 0 \\ e^{-(t-\tau)} \end{bmatrix}}{1} \cdot 1
\]

\[
= \begin{bmatrix}
\int_0^t e^{-2(t-\tau)} \,d\tau \\
\int_0^t -4e^{-2(t-\tau)} + 4e^{-(t-\tau)} \,d\tau
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{2} - \frac{1}{2}e^{-2t} \\
2 + 2e^{-2t} - 4e^{-t}
\end{bmatrix}
\]
Matlab “solution”

% Spectral decomposition
A=[-2 0; 4 -1];
B=[1 0]’;
C=eye(2);
D=0;
[V,Lambda]=eig(A);

% Verify that spectral decomposition is correct
A-V*Lambda*inv(V)

% Check that analytic solution is correct
% Step 1: Plot analytic solution
%t=0:0.001:10;
x1=exp(-2*t)+1/2-1/2*exp(-2*t);
x2=-4*exp(-2*t)+6*exp(-t)+2+2*exp(-2*t)-4*exp(-t);
plot(t,x1,’r’,t,x2,’b’)

% Step 2: Plot step response using Control Toolbox commands
x0=[1 2]’;
sys=ss(A,B,C,D);
u=ones(size(t));
figure
lsim(sys,u,t,x0)
Remarks

- Assumption $\lambda_i \neq \lambda_j$ for $i \neq j$ can be relaxed to "there exists a basis of eigenvectors".

- General case needs the notion of Jordan matrix and generalized eigenvector chain. Not difficult, but fussy.

- Output equation $y = Cx + Du$ straightforward to solve: just plug in expression for $x$ found solving the state equation, and expression for $u$ (known).
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Laplace-transform analysis of state-space systems

Laplace-transform applied componentwise to vector functions. By linearity, from state equation:

\[
L \left( \frac{d}{dt} x \right) = A \mathcal{L} (x) + B \mathcal{L} (u)
\]

\[
= : X(s) + U(s)
\]
Laplace-transform applied componentwise to vector functions. By linearity, from state equation:

\[ \mathcal{L} \left( \frac{d}{dt} x \right) = A \mathcal{L} (x) + B \mathcal{L} (u) \]

Differentiation property:

\[ sX(s) - x(0^-) = AX(s) + BU(s) \]
Laplace-transform analysis of state-space systems

Laplace-transform applied componentwise to vector functions. By linearity, from state equation:

\[ \mathcal{L} \left( \frac{d}{dt} x \right) = A \mathcal{L}(x) + B \mathcal{L}(u) \]

\[ =: X(s) \quad \quad =: U(s) \]

Differentiation property:

\[ sX(s) - x(0^-) = AX(s) + BU(s) \]

Conclude

\[ X(s) = (s I_n - A)^{-1} x(0^-) + (s I_n - A)^{-1} BU(s) \]

and

\[ Y(s) = C (s I_n - A)^{-1} x(0^-) + \left[ C (s I_n - A)^{-1} B + D \right] U(s) \]
The transfer function

By inspection easy to see that

\[(sI_n - A)^{-1} = \frac{1}{s} I_n + A \frac{1}{s^2} + \cdots + A^{k-1} \frac{1}{s^k} + \cdots\]

Substituting in previous formulas and inverse transforming by partial fraction expansion we find again time-domain solution. Only, now we can work componentwise (many problems, but at least scalar ones!)

Rational matrix

\[H(s) = C \, (sI_n - A)^{-1} \, B + D\]

is transfer function of system.
Output of a linear system is **free response** (due to the initial conditions) + **forced response** (i.e. convolution of input with impulse response)
Remarks

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Here impulse response is

\[ y(t) = \begin{cases} 
0 & , \quad t < 0 \\
D & , \quad t = 0 \\
Ce^{At} B & , \quad t > 0 
\end{cases} \]
Impulse response and transfer function

Let $m = 1$ (single input), and $u(\cdot) = \delta(\cdot)$, where $\delta(\cdot)$ is Dirac delta (impulse). Note $\mathcal{L}(\delta(\cdot)) = 1$. Then

$$Y(s) = \left( C \left( sl_n - A \right)^{-1} B + D \right) U(s) = C \left( sl_n - A \right)^{-1} B + D.$$
Impulse response and transfer function

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The transfer function is the Laplace transform of the impulse response.

For time-domain, inverse-transform the transfer function, find:

$$y(t) = \begin{cases} 
0 , & t < 0 \\
D , & t = 0 \\
Ce^{At}B , & t > 0 
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Impulse response and transfer function

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\end{cases}
\]

Generalized in natural way to multiple inputs.
Free response: the role of eigenvectors-1

For $u(\cdot) = 0$, $x(t) = e^{At} x(0^-)$. 
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Assuming basis of eigenvectors,

$$x(t) = T \text{diag}(e^{\lambda_i t})_{i=1,...,n} T^{-1} x(0^-)$$
Free response: the role of eigenvectors

For $u(\cdot) = 0$, $x(t) = e^{At}x(0^-)$. 

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Free response: the role of eigenvectors

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Assuming basis of eigenvectors,

$$
x(t) = T \text{diag}(e^{\lambda_i t})_{i=1,\ldots,n} T^{-1}x(0^-) \equiv \alpha
$$

$$
= \begin{bmatrix}
e^{\lambda_1 t} \alpha_1 \\
\vdots \\
e^{\lambda_n t} \alpha_n
\end{bmatrix}
$$
Free response: the role of eigenvectors

For $u(\cdot) = 0$, $x(t) = e^{At}x(0^-)$.

Assuming basis of eigenvectors,

$$x(t) = T \operatorname{diag}(e^{\lambda_i t})_{i=1,\ldots,n} \underbrace{T^{-1}x(0^-)}_{=:\alpha}$$

$$= T \operatorname{diag}(e^{\lambda_i t} \alpha_i)_{i=1,\ldots,n}$$

$$= \sum_{i=1}^{n} v_{\lambda_i} e^{\lambda_i t} \alpha_i$$
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Free response is linear combination of eigenvectors of $A$, associated with exponentials related to the eigenvalues of $A$, with coefficients related to the initial conditions $x(0^-)$. 
Free response: the role of eigenvectors-1

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Free response is linear combination of eigenvectors of $A$, associated with exponentials related to the eigenvalues of $A$, with coefficients related to the initial conditions $x(0^-)$. 
Free response: the role of eigenvectors-2

\[ x(t) = \sum_{i=1}^{n} v_{\lambda_i} e^{\lambda_i t} \alpha_i \]

\( v_{\lambda_i} e^{\lambda_i t} \) are the natural modes of the system.
Free response: the role of eigenvectors-2

\[ x(t) = \sum_{i=1}^{n} v_{\lambda_i} e^{\lambda_i t} \alpha_i \]

\( v_{\lambda_i} e^{\lambda_i t} \) are the natural modes of the system.

In Laplace-transform domain, \( X(s) = (sI - A)^{-1} x(0^-) \) and a parallel analysis can be performed.
$x(t) = \sum_{i=1}^{n} v_{\lambda_i} e^{\lambda_i t} \alpha_i$

$v_{\lambda_i} e^{\lambda_i t}$ are the **natural modes** of the system.

In Laplace-transform domain, $X(s) = (sI - A)^{-1} x(0^-)$ and a parallel analysis can be performed.

Natural modes important in many areas of engineering: mechanical, civil, etc.
Consider the system described by

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x .
\]

1. Identify the natural modes;
2. Find \( x(t) \) when \( x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \);
3. Find \( x(t) \) when \( x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).
4. Compare and discuss your answers to questions 2 and 3.
5. Write a Matlab script to check your answers.
Matlab hint for tutorial question 1

A=[0 1; -2 -3];
b=[0 0]’;
c=eye(2);
d=0;
sys=ss(A,b,c,d);
y1=initial(sys,[1,1]’);
y2=initial(sys,[1,-1]’);
plot(y1(:,1),’r’);
hold on;
plot(y1(:,2),’g’);
plot(y2(:,1),’b’);
plot(y2(:,2),’m’);
title(’first (red, green) and second (blue, magenta) initial condition response’);
hold off
A continuous-time system has an impulse response $h(\cdot)$ whose value at $t$ equals:

$$h(t) := \begin{cases} 
0 & \text{for } t < 0 \\
\left(-\frac{3}{2} + \frac{1}{2} \alpha\right) e^{-3t} + (2 - \alpha) e^{-2t} + \left(\frac{\alpha - 1}{2}\right) e^{-t} & \text{for } t \geq 0
\end{cases}$$

where $\alpha$ is a real number.

Find the transfer function of the system.
Consider the system described by

\[
\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x.
\]

One eigenvalue of the matrix \( A \) is -1.

Compute the transfer function of the system.

Compute the free response with \( x(0^+) = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \).
A help for exercise 3

Recall \((sl - A)^{-1} = \frac{1}{\det(sl - A)} \text{Adj}(sl - A)\); now

\[
\text{Adj}(sl - A) = \begin{pmatrix}
11 + 6s + s^2 & 6 + s & 1 \\
-6 & 6s + s^2 & s \\
-6s & -6 - 11s & s^2
\end{pmatrix};
\]

since \(C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\) and \(B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\), \(H(s)\) is the \((3, 1)\) element of \(\text{Adj}(sl - A)\) divided by \(\det(sl - A)\).
Stability, internal and external
Motivation

Stability is essential property for controlled systems.
Motivation

Stability is essential property for controlled systems. Up until now, only input-output stability considered.
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Need to:

• Formalize stability for state-space systems;
• Understand relation between state- and input-output stability.

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Need to:

- Formalize stability for state-space systems;
- Understand relation between state- and input-output stability.

Will lead to important insights.
Bounded-Input, Bounded-Output (BIBO) stability

External (i.e. input/output-based) concept
Bounded-Input, Bounded-Output (BIBO) stability

External (i.e. input/output-based) concept

System as i/o map $\mathcal{F}$: for zero initial conditions,

$$u \rightarrow y = \mathcal{F}(u)$$
$$u(t) \rightarrow \int_0^t C e^{A(t-\tau)} Bu(\tau) d\tau$$
Bounded-Input, Bounded-Output (BIBO) stability

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Stability: bounded inputs (i.e. $\|u\|_\infty < M$) produce bounded outputs (i.e. $\|y\|_\infty < M'$), where

$$\|u\|_\infty := \sup_{t \in \mathbb{R}} \|u(t)\| := \sup_{t \in \mathbb{R}} \sqrt{u(t)^\top u(t)}$$
Bounded-Input, Bounded-Output (BIBO) stability

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Theorem: The following statements are equivalent:

- $H(s)$ is BIBO-stable;
- All poles of $H(s)$ are in the open left half-plane $\mathbb{C}_-$. 
Bounded-Input, Bounded-Output (BIBO) stability

**Stability:** bounded inputs (i.e. \( \| u \|_\infty < M \)) produce bounded outputs (i.e. \( \| y \|_\infty < M' \)), where

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\]

**Theorem:** The following statements are equivalent:

- \( H(s) \) is BIBO-stable;
- All poles of \( H(s) \) are in the open left half-plane \( \mathbb{C}_- \).

Necessity, SISO case: prove by contradiction...
State-concept: if $x(0)$ close to equilibrium point 0, corresponding trajectory $x(\cdot)$ stays “close” to zero (marginal stability) or returns to it (asymptotic stability) for $t \to \infty$. 
Internal stability

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Theorem: Consider $\frac{d}{dt} x = Ax$. The following statements are equivalent:

- $x(0) = 0$ is an asymptotically stable equilibrium;
- All eigenvalues of $A$ are in the open left half-plane, i.e. they have negative real part.
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State-concept: if $x(0)$ close to equilibrium point 0, corresponding trajectory $x(\cdot)$ stays “close” to zero (marginal stability) or returns to it (asymptotic stability) for $t \to \infty$.

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- $x(0) = 0$ is an asymptotically stable equilibrium;
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Necessity: assume for simplicity diagonalizable $A$, use argument by contradiction.
Internal and input-output stability

¿Are these different stability concepts relevant for engineering?
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Internal and input-output stability

¿Are these different stability concepts relevant for engineering?

¡What happens inside the box is not necessarily mirrored by what happens at the terminals!
An example-1

Transfer function \( \frac{1}{s+1} \) is (BIBO) stable.
An example-1

Transfer function \( \frac{1}{s+1} \) is (BIBO) stable.

State-space system

\[
\frac{d}{dt}x = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
\]

has same transfer function.
An example-1

Transfer function $\frac{1}{s+1}$ is (BIBO) stable.

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$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

has same transfer function.

Solution of state-space equation is

$$x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} x(0) + \int_{0}^{t} \begin{bmatrix} e^{-(t-\tau)} \\ 0 \end{bmatrix} u(\tau) d\tau$$
An example-2

For $u(\cdot) = \bar{1}(\cdot)$, Heaviside step

\[
x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} x(0) + \begin{bmatrix} 1 - e^{-t} \\ 0 \end{bmatrix}
\]

\[
y(t) = e^{-t} x_1(0) + 1 - e^{-t}
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An example-2

For $u(\cdot) = \bar{1}(\cdot)$, Heaviside step

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$$y(t) = e^{-t}x_1(0) + 1 - e^{-t}$$

For $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, $y(t) = 1$ for all $t \geq 0$. Very good!
Bounded-input, bounded-output!
For $u(\cdot) = \bar{1}(\cdot)$, Heaviside step

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x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} x(0) + \begin{bmatrix} 1 - e^{-t} \\ 0 \end{bmatrix}
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For $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, $y(t) = 1$ for all $t \geq 0$. Very good! Bounded-input, bounded-output!

However,

\[x_1(t) = 1 \text{ and } x_2(t) = e^{2t};\]

looking at $y$ alone you’d never know $x$ is exploding!
Remarks

• Output can look good even if state explodes;
• External point of view is deficient, in view of "natural" state representations;
• This leads to the notion of minimality, in turn is related to structural properties such as controllability and observability.
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- Output can look good even if state explodes;
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- This leads to the notion of **minimality**, in turn is related to **structural properties** such as controllability and observability.
Structural properties: controllability and observability
Controllability - an example (compartmental model) - 1

C2

k_{23}

k_{12}

k_{10}

C1

k_{31}

k_{13}

C3

I

k_{12}

k_{21}
Controllability: an example (compartmental model)

$I$: drug entering body, modeled as 3 compartments (e.g. liver, blood, muscle) exchanging substances
Controllability- an example (compartmental model)-1

$I$: drug entering body, modeled as 3 compartments (e.g. liver, blood, muscle) exchanging substances

$k_{ij}$: rates at which substances are created/exchanged
Controllability- an example (compartmental model)

$I$: drug entering body, modeled as 3 compartments (e.g. liver, blood, muscle) exchanging substances

$k_{ij}$: rates at which substances are created/exchanged

State $x$ is concentration in each compartment:

$$\frac{dx}{dt} = \begin{bmatrix} -(k_{10} + k_{12} + k_{13}) & k_{21} & k_{31} \\ k_{12} & -k_{21} & 0 \\ k_{13} & 0 & -k_{31} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} I$$
Controllability- an example (compartmental model)-2

\[ \frac{d}{dt} x = \begin{bmatrix} -(k_{10} + k_{12} + k_{13}) & k_{21} & k_{31} \\ k_{12} & -k_{21} & 0 \\ k_{13} & 0 & -k_{31} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} I \]

¿Given possible to obtain arbitrary distribution of substance?
Controllability- an example (compartmental model)-2

Given

\[
\frac{d}{dt} x = \begin{bmatrix}
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k_{12} & -k_{21} & 0 \\
k_{13} & 0 & -k_{31}
\end{bmatrix} x + \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} I
\]

possible to obtain arbitrary distribution of substance?

¿From arbitrary \(x(0)\), possible to achieve, by choosing \(I(\cdot)\), arbitrary concentration in the compartments?
Controllability- an example (compartmental model)-2

Given
\[ \frac{d}{dt} x = \begin{bmatrix} -(k_{10} + k_{12} + k_{13}) & k_{21} & k_{31} \\ k_{12} & -k_{21} & 0 \\ k_{13} & 0 & -k_{31} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} I \]

possible to obtain arbitrary distribution of substance?

¿From arbitrary \( x(0) \), possible to achieve, by choosing \( I(\cdot) \), arbitrary concentration in the compartments?

Soon we’ll learn it is if and only if \( k_{12}k_{13}(k_{21} - k_{31}) \neq 0 \).
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Complex dynamical question reduced to algebra!
Controllability

\[ \frac{d}{dt} x = Ax + Bu \]

is controllable if for every \( \hat{x} \in \mathbb{R}^n \) and every \( x(0) \in \mathbb{R}^n \) there exists \( T \in \mathbb{R}_+ \) and an input function \( u(\cdot) \) s.t.

\[
x(T) = e^{AT}x(0) + \int_0^T e^{A(t-\tau)} Bu(\tau) d\tau = \hat{x}
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Controllability

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\[ x(T) = e^{AT} x(0) + \int_0^T e^{A(t-\tau)} Bu(\tau) d\tau = \hat{x} \]

Arbitrary \( \hat{x} \) and \( x(0) \iff x(0) = 0 \). 

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Controllability

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\[ x(T) = e^{AT} x(0) + \int_0^T e^{A(t-\tau)} Bu(\tau) \, d\tau = \hat{x} \]

**Theorem:** \( \frac{d}{dt} x = Ax + Bu \) is controllable if and only if the controllability matrix

\[ C(A, B) := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times m \cdot n} \]

has full row rank \( n \).
Remarks

- Proof not difficult, but a bit technical in continuous-time;
- Complex dynamical issue reduced to linear algebra question!
- Special case $m = 1$ (single-input): controllability if and only if $\det(C(A, B)) \neq 0$. 
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Exercise

In the compartmental model, set all constants equal to 1. What is the controllability matrix? What is its rank?

Repeat the same calculations with all constants equal to 1, except $k_{21} = 2$.

Matlab **command to use**: `ctrb` and `rank`. 
Determining $x(0)$ from $u(\cdot), y(\cdot)$
Determining $x(0)$ from $u(\cdot)$, $y(\cdot)$

$(A, c)$ is observable if $\forall$ $u(\cdot)$, $y(\cdot)$, there exists a unique $x(0)$ such that

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$
Observability

Determining $x(0)$ from $u(\cdot)$, $y(\cdot)$

$(A, c)$ is observable if $\forall u(\cdot), y(\cdot)$, there exists a unique $x(0)$ such that

$$y(t) = C e^{At} x(0) + \int_0^t C e^{A(t-\tau)} B u(\tau) d\tau + D u(t)$$

Equivalent with

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ cA^{n-1} \end{bmatrix} = n$$

$=: \mathcal{O}(A,C)$
Remarks

- Proof not difficult, but a bit technical in continuous-time;
  - Complex dynamical issue reduced to linear algebra question!
  - Special case $m = 1$ (single-output): observability if and only if $\det(O(A, C)) \neq 0$. 
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Example-1

Circuit with voltage source $u$, and current $y$.
Example-1

Circuit with voltage source $u$, and current $y$.

State equations are

$$\frac{d}{dt} \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_C \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1} \\ 0 & 0 & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_C \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$
Example-1

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\]

\[
y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x
\]

Transfer function is $G(s) = \frac{C(L_1+L_2)s}{CL_1L_2s^2-(L_1+L_2)}$.

Funny, for a 3rd order system...
¿Can we determine $x(0)$ observing $u(\cdot)$ and $y(\cdot)$?
Example-2

¿Can we determine \( x(0) \) observing \( u(\cdot) \) and \( y(\cdot) \)?

Assume \( C = 1 \) \( F \), \( L_1 = 2 \) \( H \) \( L_2 = 2 \) \( H \). Compute

\[
O(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -2 \\ -2 & -2 & 0 \end{bmatrix}
\]
Example-2

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First two columns are the same: not full rank!
¿Can we determine $x(0)$ observing $u(\cdot)$ and $y(\cdot)$?

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First two columns are the same: not full rank!

This means that $x(0) = 0$ and $x(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ give rise to the same $y(\cdot)$, no matter what $u(\cdot)$ is applied.
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First two columns are the same: not full rank!

This means that $x(0) = 0$ and $x(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ give rise to the same $y(\cdot)$, no matter what $u(\cdot)$ is applied.

Not surprising- any current around inductor loop does not influence behavior at external terminals $u$ and $y$.  
Remarks on controllability/observability

- **Controllability**: ability to steer the state of the system wherever we want;

- Observability: less intuitive, but often state is impossible or too costly to measure directly, so ability to infer it from input and output sequences is crucial;

- Structural properties: they depend on $A$, $B$, $C$ and the relation between them in a geometric sense (ranks of controllability/observability matrices).
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- **Structural** properties: they depend on $A$, $B$, $C$ and the relation between them in a geometric sense (ranks of controllability/observability matrices).
For checking observability:
\[ O = \text{obsv}(A, C) \]
\[ \text{rank}(O) == \text{size}(A, 2) \]
If answer is “1” then system is observable.

For controllability, use \texttt{ctrb}(A, b) instead.
Tutorial exercise 4

Consider the system

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \alpha & 0 \\ 1 & 2 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x
\]

where \( \alpha \) is a real parameter.

- Determine the \( \alpha \)s for which system is observable.
- For the \( \alpha \)s for which the system is not observable, prove that \( x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( x'(0) = x(0) + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \) cannot be distinguished from each other observing a given \( u(\cdot) \) and the corresponding \( y(\cdot), y'(\cdot) \).
- Verify the previous point writing a Matlab program.
Matlab hint for tutorial question 4

\[ A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 2 & 2 \end{bmatrix}; \]

\[ b = [1 \ 0 \ 1]'; \]

\[ c = [0 \ 0 \ 1]; \]

\[ d = 0; \]

\[ \text{sys}=\text{ss}(A,b,c,d); \]

\[ y_1=\text{initial(sys},[1 \ 1 \ 1]'); \]

\[ y_2=\text{initial(sys},[1 \ 1 \ 1]'+[-2 \ 1 \ 0]'); \]

\[ \text{plot}(y_1-y_2) \]
Minimality
Nonuniqueness of state-space realizations

There are many ways to represent in state-space a given transfer function. The choice depends on the application.
Nonuniqueness of state-space realizations

There are many ways to represent in state-space a given transfer function. The choice depends on the application.

For example:

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 3 \\ 1 \end{bmatrix} x
\]

and

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x
\]

represent the same transfer function.
Nonuniqueness of state-space realizations

There are many ways to represent in state-space a given transfer function. The choice depends on the application.

...but also

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u
\]

\[y = \begin{bmatrix} 0 & 1 \end{bmatrix} x\]

and

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u
\]

\[y = \begin{bmatrix} 1 & 0 \end{bmatrix} x\]

and so forth.

There are 3rd (or 10\(^6\)th-) order representations!
Nonuniqueness of state-space realizations

There are many ways to represent in state-space a given transfer function. The choice depends on the application.

¿...how “non unique” is a realization?

...leads to the issue of minimality
Minimality

$(A, B, C, D)$ is minimal state representation of $H(s)$ if the state dimension is minimal among all realizations.
Minimality

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**Theorem:** The following conditions are equivalent:

- $(A, B, C, D)$ is a minimal realization of $H(s)$;

- $H(s) = C(sI - A)^{-1}B + D$, and moreover $(A, B)$ is controllable and $(C, A)$ is observable;
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- (For SISO systems) $H(s) = C(sI - A)^{-1}B + D$ is irreducible, i.e. numerator and denominator are coprime.
Minimality

\((A, B, C, D)\) is \textbf{minimal} state representation of \(H(s)\) if the state dimension is minimal among all realizations.

**Theorem**: The following conditions are equivalent

- \((A, B, C, D)\) is a minimal realization of \(H(s)\);
- \(H(s) = C(sI - A)^{-1}B + D\), and moreover \((A, B)\) is controllable and \((C, A)\) is observable;
- (For SISO systems) \(H(s) = C(sI - A)^{-1}B + D\) is irreducible, i.e. numerator and denominator are coprime.
- (For SISO systems) The dimension of \(A\) equals the degree of the denominator of \(H(s)\) (assumed irreducible).
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Theorem: The following conditions are equivalent

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• (For SISO systems) $H(s) = C(sI - A)^{-1}B + D$ is irreducible, i.e. numerator and denominator are coprime.

• (For SISO systems) The dimension of $A$ equals the degree of the denominator of $H(s)$ (assumed irreducible).

Minimality $\iff$ controllability and observability $\iff$ (SISO) irreducibility
The Kalman decomposition

State space (system) consists of 4 parts:
- controllable and observable;
- controllable and non-observable;
- non-controllable and observable;
- non-controllable and non-observable

Only the controllable and observable part represent faithfully the input-output relation
Remarks on the Kalman decomposition

- A systems, not a control tool: it only shows the system as it is;
- May lead to re-design, but then new system is not the old one!
BIBO- and internal stability

If state representation is minimal,

BIBO stability $\iff$ internal stability
BIBO- and internal stability

If state representation is minimal,

\[ \text{BIBO stability} \iff \text{internal stability} \]

If state representation is not minimal,

\[ \text{BIBO stability} \Leftarrow \text{internal stability} \]
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If state representation is not minimal,

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*Proof*: based on the fact that

\[ \{ \text{Poles of } C(sI - A)^{-1}B \} \subseteq \{ \text{Eigenvalues of } A \} \]
BIBO- and internal stability

If state representation is minimal,

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¡To “fully” stabilize a system, we need to modify the eigenvalues of $A$, not only the poles of $H(s)$!
BIBO- and internal stability

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¡To “fully” stabilize a system, we need to modify the eigenvalues of \( A \), not only the poles of \( H(s) \)!

The main reason for doing state feedback!
Theorem: if \((A_1, B_1, C_1, D_1)\) and \((A_2, B_2, C_2, D_2)\) are minimal realizations of the same transfer function, then there exists a nonsingular matrix \(T\) such that

\[
\begin{align*}
A_1 &= T^{-1} A_2 T \\
B_1 &= T^{-1} B_2 \\
C_1 &= C_2 T \\
D_1 &= D_2
\end{align*}
\]
Consider again the system

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \alpha & 0 \\ 1 & 2 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x
\]

where \( \alpha \) is a real parameter.

Compute the transfer function of the system \( H_\alpha(s) \) (in general, it depends on \( \alpha \)).

Compare the degree of the denominator of \( H_\alpha(s) \) with the dimension of the realization.
Observe that

$$(sl - A)^{-1} = \frac{1}{(s - \alpha)(s^2 - 2s - 1)} \begin{bmatrix}
(s - 2)(s - \alpha) & s & s - \alpha \\
0 & s^2 - 2s - 1 & 0 \\
(s - \alpha) & 2s + 1 & s(s - \alpha)
\end{bmatrix}$$
Solution

Observe that

\[
(sI - A)^{-1} = \frac{1}{(s - \alpha)(s^2 - 2s - 1)} \begin{bmatrix}
(s - 2)(s - \alpha) & s & s - \alpha \\
0 & s - \alpha & 2s + 1 \\
(s - 2) & s^2 - 2s - 1 & s - \alpha
\end{bmatrix}
\]

Now

\[
c(sI - A)^{-1} = \frac{1}{(s - \alpha)(s^2 - 2s - 1)} \begin{bmatrix}
s - \alpha & 2s + 1 & s(s - \alpha)
\end{bmatrix}
\]
Solution

Observe that

\[(sl - A)^{-1} = \frac{1}{(s - \alpha)(s^2 - 2s - 1)} \begin{bmatrix} (s - 2)(s - \alpha) & s & s - \alpha \\ 0 & s^2 - 2s - 1 & 0 \\ s - \alpha & 2s + 1 & s(s - \alpha) \end{bmatrix} \]

Now

\[c(sl - A)^{-1} = \frac{1}{(s - \alpha)(s^2 - 2s - 1)} \begin{bmatrix} s - \alpha & 2s + 1 & s(s - \alpha) \end{bmatrix}\]

and consequently

\[c(sl - A)^{-1}b = \frac{(1 + s)(s + \alpha)}{(s + \alpha)(-1 - 2s + s^2)} \]

\[= \frac{(1 + s)}{(-1 - 2s + s^2)} \]
Solution

Observe that

\[
(sI - A)^{-1} = \frac{1}{(s - \alpha)(s^2 - 2s - 1)} \begin{bmatrix}
(s - 2)(s - \alpha) & s & s - \alpha \\
0 & s^2 - 2s - 1 & 2s + 1 \\
s - \alpha & s(s - \alpha) & 0
\end{bmatrix}
\]

Now

\[
c(sI - A)^{-1} = \frac{1}{(s - \alpha)(s^2 - 2s - 1)} \begin{bmatrix}
s - \alpha & 2s + 1 & s(s - \alpha)
\end{bmatrix}
\]

and consequently

\[
c(sI - A)^{-1}b = \frac{(1 + s)(s + \alpha)}{(s + \alpha)(-1 - 2s + s^2)} = \frac{(1 + s)}{(-1 - 2s + s^2)}
\]

Simplification between numerator and denominator occurs no matter what \( \alpha \) is. Even if the system is observable \( (\alpha \neq -\frac{1}{2}) \), it is never controllable. Can’t be minimal!
Consider the system described in input-state-output equations by

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 3 & 1 \end{bmatrix} x.
\]

- Is the representation observable?
- Is the representation controllable?
- Compute the transfer function of the system.
- Is the representation minimal?
- Compute a different representation of the same transfer function.