1 Introduction

The coordination of groups of mobile autonomous agents and their synchronization is an active area of theoretical research at the crossroads of physics, discrete- and continuous mathematics, and control theory. Such problem arises in areas as different to each other as biology, computer animation, robotics, power systems, and many others. An accessible introduction to such research area is [2], which you are recommended to read in order to understand the context and identify some of the most important issues facing control theorists and engineers.

2 Aims

The aims of this reading assignment are:

• to introduce you to the topic of consensus and cooperation;

• to make you read in depth one research paper about consensus and cooperation;

• to have you solve a simple consensus problem by applying some mathematical results and by implementing via Matlab a simple simulation.

What is required of you in this assignment is:

• elementary knowledge about linear systems and stability (that obtained through attending the ELEC6243 lectures);

• elementary knowledge of Matlab (see the ELEC6243 lectures and the help command).

3 Reading

Please download through the University of Southampton Library website the paper [1]. At the beginning, read carefully sections 1–3 at least.

Some of the questions below are meant to make it easier for you to understand the mathematical control theory concepts and to use them confidently.
4 Questions

Answer the following questions, justifying your answers as much as possible.

Question 1: Consider the graph depicted in Fig. 1. Assume the weights $a_{ij}$ of the edges are all equal to 1. The following questions use the notation of section II of [1], and the indexing of the vertices shown in Fig. 1.

Figure 1: Network structure

Question 1.1 Identify the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$.

Question 1.2 Write down the adjacency matrix $A$ for the graph.

Question 1.3 Write down the Laplacian matrix $L$ for the graph.

Question 1.4 Calculate the eigenvalues of $L$ using Matlab.

Question 1.5 Identify the neighbors sets $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_4$.

Question 1.6 Calculate the left eigenvector $w_l^\top$ of $L$ associated with the zero eigenvalue, i.e. the vector $w_l^\top$ such that $w_l^\top L = 0$.

(4 marks)

Question 2: Consider a system with dynamics described by $\frac{d}{dt} x = Ax + Bu$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$  

Use the state-feedback technique in Remark 4 p. 4461 of [1] to transform such system to the state space description $\frac{d}{dt} x = A'x + B'u$ of a third-order integrator.

(2 marks)

In the following $A'$ and $B'$ are the matrices obtained answering Question 2. They represent the dynamics of each agent situated in each node of the graph in Fig. 1. Such agents exchange information, consisting of the values of their states, with their neighbours. Each agent evaluates such information via
a so-called protocol, and adjusts its own dynamics accordingly. The objective of the control engineer is to design a protocol, the same for each agent, that brings each agent to the same asymptotic state. In the following questions we design a consensus protocol according to the design procedure of [1], simulate the corresponding network, and verify that each agent asymptotically achieves the same state value.

**Question 3:** Consider the consensus protocol in formula (2) p. 4459 of [1], and the dynamics (3) on p. 4460.

**Question 3.1** What is the value of $m$?

**Question 3.2** Write down the closed-loop state-space description (3) of $\frac{d}{dt}x = A'x + B'u$ under the consensus protocol (2), with $c_k$ and $k_i$ symbolic, for the agent in vertex 1 and for the agent in vertex 2.

(3 marks)

**Question 4:** The Kronecker product $A \otimes B$ of the matrices $A = [a_{ij}]$ $(m \times n)$ and $B = [b_{ij}]$ $(p \times r)$- note that $A$ and $B$ have not necessarily compatible dimensions!- is defined by

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{12}B & \ldots & a_{mn}B \end{bmatrix},$$

and it is a $mp \times nr$ matrix. The dynamics of a network of identical agents is much easier to describe using Kronecker products, see Question 4.2.

**Question 4.1:** Define $A := \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $B := \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Compute $A \otimes B$.

**Question 4.2:** The closed-loop dynamics of the whole network are described by the equation (4) p. 4460 of [1]. Define $k_i = 1$, $i = 1, \ldots, 4$, so that $\mathcal{Y} = I_4$. Write the closed-loop matrix $\Omega$ for the choice of parameters $c_1 = 2$, $c_2 = 3$.

(4 marks)

**Question 5:** Using the values of the eigenvalues of $L$ found answering Question 1.4, write down the coefficients of the polynomial defined in formula (5) p. of [1]. Verify that the eigenvalues of $\Omega$ are the roots of this polynomial. Verify that they are all in the open left-hand plane, except for one at zero.

(3 marks)
Question 6: Let $\Omega$ be a $n \times n$ matrix with distinct eigenvalues. Assume that one eigenvalue equals zero, and all the others are in the open left-hand plane. Consider the system described by

$$\frac{d}{dt} x = \Omega x,$$

with initial condition $x(0) =: \bar{x}$. Prove that $\lim_{t \to \infty} x(t)$ exists, and that it is either zero, or it lies in the direction of the eigenvector of $\Omega$ associated with the zero eigenvalue.

(6 marks)

Question 7: Write a Matlab program that simulates the whole network, under the consensus protocol specified in Question 4.2.

The initial conditions of each agents must be set to random values. Your program must print out the average such random values, which are needed to answer Question 8 below.

Such program should also plot the graph of the evolution of the state variables $x_k$ of all the agents, $k = 1, 2, 3$ in three separate graphs.

(5 marks)

Question 8: Use the recorded values for the initial conditions of the initial states to verify that the consensus state values, i.e. the asymptotic values of the state variables of each agent, satisfy the result of Th. 1 p. 4460 of [1].

(3 marks)

References
