Answer ALL questions in Section A, TWO questions from Section B and ONE question from Section C.

Section A carries 25 out of the 100 total marks for the exam paper and you should aim to spend about 30 minutes on it. Section B carries 50 out of the 100 total marks for the exam paper and you should aim to spend about 60 minutes on it. Section C carries 25 out of the 100 total marks for the exam paper and you should aim to spend about 30 minutes on it.

An approximate marking scheme is indicated to the right of each question.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Fully working SystemVerilog code is not required and no marks will be deducted for presenting partial code which clearly explains your implementation.

Only University approved calculators may be used.
Question 1 Digital Systems
Produce a logic circuit that implements the following SystemVerilog description:

```verilog
module circuit1 (input logic clk, A, output logic B, C, D);
    always_ff @(posedge clk)
        if (A)
            C <= B;
    always_ff @(posedge clk)
        D <= C;
    always_ff @(posedge clk)
        if (A)
            B <= B ^ C ^ D;
        else B <= 1'b0;
endmodule
```

(Figure 1)
Question 2 Digital Systems

Develop a SystemVerilog module to implement the state machine defined by the state transition table below. The machine has four states S0, S1, S2, and S3, one input A and one output Y. State if this is a Mealy or a Moore machine and explain your answer.

<table>
<thead>
<tr>
<th>A</th>
<th>Present State</th>
<th>Next State</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>S0</td>
<td>S1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>S0</td>
<td>S0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>S1</td>
<td>S1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>S1</td>
<td>S2</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>S2</td>
<td>S1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>S2</td>
<td>S3</td>
<td>0</td>
</tr>
<tr>
<td>X</td>
<td>S3</td>
<td>S3</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 1:** State transition table for Question 2.

(5 marks)

```verilog
module q2a (input logic clk, A, output logic Y);

// use enum to declare state variables present, next [2 marks]
enum logic [1:0] { S0, S1, S2, S3} present, next;

// use separate always_ff and always_com [1 mark]
always_ff @ ( posedge clk)
    state <= next;

always_com // next logic, 1 mark
    case(state)
        S0: next = (A? S1: S0);
        S1: next = (A? S2: S1);
        S2: next = (A? S3: S1);
        S3: next = S3;

always_com // Y logic, Moore machine: 1 mark
    case(state)
        S0: Y = 0;
        S1: Y = 1;
        S2: Y = 0;
        S3: Y = 1;

endmodule // module q2a
```

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Question 3 Signal Processing

$v[n]$ is a speech signal, the PDF of which is given by

$$p_V(v) = \frac{1}{2} e^{-\alpha |v|}$$

(a) Determine $\alpha$.

(b) Determine and sketch the CDF $P_V(v)$ corresponding to the PDF of $v[n]$.

(5 marks)

(a) Condition:

$$\int_{-\infty}^{\infty} p_V(v) dv = 1.$$  

$$\int_{-\infty}^{\infty} \frac{1}{2} e^{-\alpha |v|} dv = \int_{0}^{\infty} e^{-\alpha v} dv = \left[-\alpha e^{-\alpha v}\right]_{0}^{\infty} = \alpha$$  \hspace{1cm} (1)

Hence $\alpha = 1$.

(b) CDF:

$$P_V(v) = \int_{-\infty}^{v} p_V(u) du$$  \hspace{1cm} (2)

$$v < 0 : \quad P_V(v) = \int_{-\infty}^{v} \frac{1}{2} e^{-u} du = \frac{1}{2} [e^{-u}]_{-\infty}^{v} = \frac{1}{2} e^{-v} ;$$  \hspace{1cm} (3)

$$v \geq 0 : \quad P_V(v) = \frac{1}{2} + \int_{0}^{v} e^{-u} du = \frac{1}{2} + \left[ -e^{-u} \right]_{0}^{v} =$$  

$$= \frac{1}{2} (1 - e^{-v} + 1) = 1 - \frac{1}{2} e^{-v} ;$$  \hspace{1cm} (5)

Sketch:
Question 4 Signal Processing

Briefly describe how a linear $b$-bit Pulse Code Modulation (PCM) scheme operates and comment on the quantisation noise. Also, carefully draw and label its schematic and quantiser characteristic. (5 marks)

- In linear PCM, the quantiser approximates a straight line with $2^R$ levels; example with $R = 3$ and asymmetric characteristic.

- the input signal has to use the range $[-V; +V]$ as best as possible, otherwise clipping or coarse resolution result;

- quantisation noise: maximum $q/2$, uniformly distributed, additive;

![Schematic of a linear PCM]

![Quantiser characteristic graph]
Question 5 Signal Processing

Figure 2 shows the block diagram of a half-rate convolutional encoder, where \( b_k \) is the \( k \)th information bit, while \( c_k^{(0)} \) and \( c_k^{(1)} \) are the corresponding two encoded bits at time instant \( k \). Complete both the coding table in Table 2 and the Trellis diagram shown in Figure 3.

\[ (5 \text{ marks}) \]

**Figure 2:** Block diagram of a unit-memory half-rate convolutional encoder, where \( S \) is the shift register content.

**Table 2:** Incomplete coding table, where \( S \) and \( S^{+1} \) are the current and next shift register contents, respectively.

<table>
<thead>
<tr>
<th>( b_k )</th>
<th>( S )</th>
<th>( c_k^{(0)} )</th>
<th>( c_k^{(1)} )</th>
<th>( S^{+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>?</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>?</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

**Figure 3:** Partial Trellis diagram.

The completed coding table is shown below:

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The completed Trellis diagram is shown below:

\[ b_k/ c_k^{(0)}/ c_k^{(1)} \]

\[ b_k = 0 \]
\[ b_k = 1 \]

**Completed Trellis diagram.**
Question 6  Digital Systems
A synthesisable SystemVerilog module is required for a sequential system that implements a 12-bit register with one 12-bit logic output port Y, four single-bit logic input ports E,R,L and CLK and one 12-bit logic input port X. The module has the following specification. If E=1'b1, the output Y increments on the next rising clock edge, i.e. Y := Y + 1; Otherwise, if the input L=1'b1, the machine loads the input X into Y, i.e. Y := X. The signal R is an active-high synchronous reset and CLK represents the clock.

(a) Develop a SystemVerilog description of the module.  

(b) Develop a testbench which will test the following scenario. On the first edge of the clock the register is reset and the reset signal remains asserted for the next 3 clock periods. For the next 5 clock periods the register Y increments. Then the value of 3 is written into the register Y, followed by an increment.

(a) [10 marks: ]

```verilog
module pc (input logic clk, R, E, L, input logic [11:0] X, output logic [11 : 0]Y );
always_ff @ ( posedge clk)
    if (R) // sync reset must appear first in the if conditions
        Y <= 12b0;
    else if (E) // increment must appear 2nd according to spec above
        Y <=Y + 1'b1;
    else if (L) // parallel load
        Y <= X;
endmodule // module pc
```

(b) [15 marks: ] Testbench:

```verilog
module pctest;
logic clk, R, E, L;
logic [11:0] X,Y;
```
pc pc1 (.*); // create an instance of the tested module

initial // clk period 10ns
begin
    clk = 0;
    forever #5 clk = ~clk;
end

initial
begin
    R = 1; // this will reset pc on 1st clock tick at 5ns
    X = 0, E=0; L = 0;
    // hold R=1 for 3 clock ticks, i.e. 15, 25, 35 ns.
    #40 E= 1; R = 0; // increments will follow
    // wait next 5 ticks, (45, 55, 65, 75, 85) i.e 50ns
    #50 E=0; L=1; X =3; // write 3 into pc
    #50 E=1, L=0; // increment
end
Question 7  Digital Systems  The following SystemVerilog module describes a sequential logic system that implements Euclid’s algorithm to compute the greatest common divisor (GCD) of two 8-bit unsigned numbers A and B, where $A \geq B$. The system begins calculations when Start becomes $1'b1$ for the duration of least one clock period and then Start becomes $1'b0$. When the algorithm finishes the output Ready becomes $1'b1$ and the 8-bit logic output GCDivisor represents the answer.

```verilog
module GCD (input logic clk, Start, input logic [7:0] A,B, output logic [7:0] GCDivisor, output logic Ready);

logic [7:0] a,b;

always_ff @(posedge clk)
  if (Start == 1'b1)
    begin
      a <= A;  b <= B;  Ready <= 1'b0;  GCDivisor = 8'h00;
    end
  else if (b == 8'h00)
    begin
      GCDivisor <= a; Ready <= 1;
    end
  else if (a > b)
    a <= a - b;
  else
    b <= b - a;
endmodule
```
(a) Complete the timing diagram below as the module computes the GCD of 15 and 10. Use the hexadecimal or decimal notation to represent the values of 8-bit waveforms at each clock period.

![Timing Diagram](image)

Figure 4: Greatest common divisor waveforms.

(15 marks)

(b) Develop a testbench in SystemVerilog that will test the above computation of the GCD where A=15 and B=10.

(10 marks)

(a) [20 marks:]

(b) [10 marks:]

```verilog
module testbench_q7b
  logic clk, Start;
  logic [7:0] A, B, GCDivisor;

  GCDivisor(.*);

  initial // clock waveform, period 10
  begin
    clk = 1'b0;
    forever #5 clk = ~clk;
  end

  initial
```

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begin
    Start = 1'b0;
    A = 15;
    B = 10;
    #7 Start = 1'b1;
    #15 Start = 1'b0; // keep Start on for clock periods, then remove
end

endmodule testbench_q7b
Question 8

(a) Explain, briefly, the advantages and disadvantages of Scan-In, Scan-Out (SISO).

(5 marks)

(b) Figure 6 shows a simple state machine, with a single input, A, a single output, S and two state variables, S and T.

Consider just the combinational part of the circuit (shown within the dashed lines).

Write a test for J/1 (J stuck at 1), in terms of S, T and A. What is the value of \( S^+ \) if the fault, J/1, does not exist? What is the value of \( S^+ \) if the fault does exist?

What other faults in the combinational part of the circuit (i.e. observable at \( S^+ \) and \( T^+ \)) are covered by this test pattern? (Note that \( \bar{S} \) and
\( \overline{T} \) must be the inverse of \( S \) and \( T \), respectively.

(10 marks)

(c) Explain how a scan-path can be included within the circuit of Figure 6 such that only one additional input is required.

How would you test the scan path to ensure that it contains no faults?

Explain how the test that you derived in part (b) can be applied to the full, sequential circuit. What would be observed at the output of the scan-path if the fault \( J/1 \) does not exist, and also if it does exist?

(10 marks)

a) Advantages: A sequential system is completely controllable and observable. Test generation is reduced to combinatorial test generation.

Disadvantages: Bigger circuit; more delays; more i/o pins.

b) To test for \( J/1 \) implies that \( J=0 \). Therefore \( A=T=1 \). To transmit the value to \( S^+ \) requires that \( K=1 \). Therefore \( S \) or \( \overline{T} \) or both must be 1. As \( T \) is already 1, \( \overline{T} \) is 0 and therefore \( S \) must be 1.

In the fault-free case, \( S^+ \) is 1, if faulty, 0.

This test will also cover \( S^+/0, T^+/0, S/0, T/0, A/0 \) and \( M/1 \) (but not \( K/0, L/0, T/1 \) or \( \overline{S}/1 \)).

c) Let’s assume that the additional input is control signal, \( B \), which enables the scan path when 1. In scan mode, \( A \) is connected to the scan i/p of \( T \); \( T \) is connected to \( S \) and the scan data output is \( S \).

Test the scan path by applying setting \( B \) to 0 and applying a sequence, 0, 1, 0, 1 to \( A \). We should see the same sequence at \( S \), delayed by 2 clock cycles.

So, to set \( S \) and \( T \) to 1, we set \( B \) to 1, \( A \) to 1 and clock twice.

Then, with \( A \) at 1, set \( B \) to 0, clock once. This loads \( T^+ \) and \( S^+ \) into the scan chain. We can observe \( S^+ \) at this point. If \( J/1 \) does not exist, \( S \) is 1, otherwise \( S \) is 0.
Section C

Question 9

(a) A random process is given by \( X(t) = A \cos(\omega t) + B \sin(\omega t) \), where \( A \) and \( B \) are independent zero-mean random variables.

(i) Find the mean \( \mu_X(t) \) of \( X(t) \).
(ii) Find the autocorrelation function \( R_{XX}(t_1, t_2) \) of \( X(t) \).
(iii) Under what conditions is \( X(t) \) wide sense stationary (WSS)?

Hint:

(i) Mean and variance
\[
\mu_x = \int_{-\infty}^{\infty} x \, p_X(x) \, dx ; \quad \sigma_x^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 \, p_X(x) \, dx
\]

(ii) Autocorrelation Function
\[
R_{XX}(t_1, t_2) = E[X(t_1) \cdot X(t_2)]
\]

(b) How do you avoid aliasing in an Analogue-to-Digital Convertor (ADC), if the signal’s frequency content is not known?

(c) A DVD-audio player supports a bit depth of 24 bits and a sampling rate of 176.4 kHz.

(i) Determine the dynamic range of 24-bit linear PCM expressed in dB.
(ii) Comment on the maximum required bandwidth of the anti-aliasing low-pass filter.
(iii) Quantify the bit-rate required for the transmission of uncompressed DVD-quality linear PCM-encoded music based on a sampling rate of 176.4 kHz and a bit depth of 24 bits.

(6 marks)

(d) Various source coding schemes will be used to encode an information source that has four possible symbols, \( \{ x_0, x_1, x_2, x_3 \} \). The set of probabilities \( \{ p_i \} \) for all source symbols \( \{ x_i \} \) as well as the corresponding Fixed Length Code (FLC) codewords \( \{ c_i^{\text{FLC}} \} \) and Huffman (Huff) codewords \( \{ c_i^{\text{Huff}} \} \) are shown in Table 3.

<table>
<thead>
<tr>
<th>Source symbols</th>
<th>( p_i )</th>
<th>( c_i^{\text{FLC}} )</th>
<th>( c_i^{\text{Huff}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 )</td>
<td>0.4</td>
<td>00</td>
<td>1</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0.3</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0.2</td>
<td>10</td>
<td>001</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0.1</td>
<td>11</td>
<td>000</td>
</tr>
</tbody>
</table>

Table 3: The probability distribution and codewords of a four-symbol source.

(i) Find the amount of information \( k_i \) that is conveyed by each source symbol \( x_i \).

(ii) What is the entropy \( H \) of the source?

(iii) What are the average codeword lengths of FLC and Huff?

(iv) What are the coding efficiencies \( R \) associated with \( c_i^{\text{FLC}} \) and \( c_i^{\text{Huff}} \)?

(v) Draw a binary tree for the Huffman codewords and then use it to find the corresponding sequence of source symbols when the Huffman-encoded bit sequence is given by 000011.

(vi) If a sequence of four source symbols taken from Table 3 have been encoded using Arithmetic code and the encoded bit sequence is given by 10111. Find out the sequence of the four source symbols.

(9 marks)
(a) $X(t) = A \cos(\omega t) + B \sin(\omega t)$

$E[A] = 0.$

$E[B] = 0.$

(i) [2 marks] 
$\mu_X(t) = E\{X(t)\} = E\{A \cos(\omega t) + B \sin(\omega t)\} = E\{A\} \cos(\omega t) + E\{B\} \sin(\omega t) = 0$

(ii) [3 marks] 
$\rho_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{(A \cos(\omega t_1) + B \sin(\omega t_1))(A \cos(\omega t_2) + B \sin(\omega t_2))\}$

$= E\{A^2\} \cos(\omega t_1) \cos(\omega t_2) + E\{B^2\} \sin(\omega t_1) \sin(\omega t_2) + E\{AB\} \cos(\omega t_1) \sin(\omega t_2) + E\{AB\} \sin(\omega t_1) \cos(\omega t_2)$

$= E\{A^2\} \cos(\omega t_1) \cos(\omega t_2) + E\{B^2\} \sin(\omega t_1) \sin(\omega t_2) + 0 + 0$

$= E\{A^2\} \frac{1}{2} (\cos(\omega t_1 - \omega t_2) + \cos(\omega t_1 + \omega t_2)) + E\{B^2\} \frac{1}{2} (\cos(\omega t_1 - \omega t_2) - \cos(\omega t_1 + \omega t_2))$

$= E\{A^2\} \frac{1}{2} \cos(\omega (t_1 - t_2)) + E\{B^2\} \frac{1}{2} \cos(\omega (t_1 + t_2))$

(iii) [3 items] 
$X(t)$ will be WSS if $E\{A^2\} = E\{B^2\}$ so that the $\rho_{XX}(t_1, t_2) = E\{A^2\} \frac{1}{2} \cos(\omega (t_1 - t_2)) + 0$

and hence the auto-correlation function is a function of the lag $t_1 - t_2$.

(b) [2 marks] An anti-aliasing filter can be used in an ADC to remove all frequency components above half sampling frequency ($f_s/2$), in order to avoid aliasing.

(c) [6 marks] A DVD-audio supports a bit depth of 24 bits and a sampling rate of 176.4 kHz.

(i) [2 marks] The dynamic range of a 24-bit linear PCM is $24 \times 6.02 = 144.48$ dB.

(ii) [2 marks] According to the Nyquist sampling theorem, we have $F_s \geq 2B_{max}$, where $B_{max} \leq F_s/2$ is the maximum bandwidth of the anti-aliasing lowpass filter. At a sampling rate of $F_s = 176.4$ kHz, the maximum bandwidth of the anti-aliasing lowpass filter is $B_{max} \leq F_s/2 = 88.2$ kHz (or 88 kHz is adequate).

(iii) [2 marks] The bitrate required is $F_s \times R = 176.4 \times 10^3 \times 24 = 4.2336 \times 10^6$ bit/s.

(d) [9 marks] 

<table>
<thead>
<tr>
<th>Source symbols</th>
<th>$p_i$</th>
<th>$c^{FLC}_i$</th>
<th>$c^{Huff}_i$</th>
<th>$k_i = -\log_2(p_i)$</th>
<th>$p_i \cdot k_i$</th>
<th>$H^{FLC}_i$</th>
<th>$p_i \cdot H^{Huff}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>0.4</td>
<td>100</td>
<td>1</td>
<td>1.3219</td>
<td>0.5288</td>
<td>1</td>
<td>0.4</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0.3</td>
<td>100</td>
<td>0</td>
<td>1.7370</td>
<td>0.5211</td>
<td>2</td>
<td>0.6</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.2</td>
<td>100</td>
<td>0</td>
<td>2.3219</td>
<td>0.4644</td>
<td>3</td>
<td>0.6</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0.1</td>
<td>100</td>
<td>0</td>
<td>3.3219</td>
<td>0.3322</td>
<td>3</td>
<td>0.3</td>
</tr>
</tbody>
</table>

(i) [1 mark] The amount of information $k_i$ that is conveyed by each source symbol $x_i$ is given in the 5th column of the above table.

(ii) [1 mark] The source entropy is given by $H = \sum_{i=0}^{3} p_i \cdot k_i = 1.8465$.

(iii) [1 mark] The average codeword lengths are $L^{FLC} = 2$ and $L^{Huff} = \sum_{i=0}^{3} p_i \cdot H^{Huff}_i = 1.90$. 

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(iv) [1 mark] The coding efficiencies are $R_{FLC} = H/L_{FLC} = 1.8465/2 = 0.9233$ and $R_{Huff} = H/L_{Huff} = 1.8465/1.9 = 0.9718$.

(v) [1 mark] The binary tree for the Huffman codewords is given below. For the Huffman coded sequence [000 01 1 1], the corresponding source sequence is [x3 x1 x0 x0].

(vi) [4 marks] If a sequence of four source symbols taken from Table 3 have been encoded using Arithmetic code and the encoded bit sequence is given by 1011. Then the equivalent decimal fraction is given by: $0.1011_b = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 0.71875$. The four source symbols are $[x_2 x_0 x_0 x_1]$ as seen from the following diagram where 0.71875 is within the range [0.7128 0.7224).
Question 10

(a) The output of a midrise uniform quantiser of input range ±5 V is encoded using 4 bits/sample with the aid of linear Pulse Coded Modulation (PCM).

(i) Determine the maximum quantisation error.

(ii) Determine the quantisation error associated with an input sample of value $s = -2.4$ V.

(iii) Determine the quantisation noise power, given that the quantisation noise is uniformly distributed.

(8 marks)

(b) The uncorrelated discrete-time random signal $x[\cdot]$ has the probability density function (PDF) of

$$p(x) = \begin{cases} 1 & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

(i) Write an expression for the cumulative distribution function (CDF) of $x[n]$.

(ii) What is the mean value of $x[n]$?

(iii) Assume that the signal $x[\cdot]$ is input to a linear system with impulse response

$$h[k] := \begin{cases} 1 & \text{for } 0 \leq k \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

What is the mean of the system output $y[\cdot]$ corresponding to the input $x[\cdot]$?

Hint: $\mu_y = \mu_x \sum_{n=-\infty}^{+\infty} h[n]$, \quad $r_{yy}[\tau] = r_{xx}[\tau] * h[\tau] * h[-\tau]$

(8 marks)
(c) The capacity of an Additive White Gaussian Noise (AWGN) channel is given by:

\[ C = B \log_2(1 + SNR) \]

where \( B \) is the passband bandwidth and \( SNR \) is the signal to noise ratio, while the transmission rate is given by \( R_s \) [symbols/s]. Assuming ideal low pass filters are used for your transmit and receive filters, while a channel code of rate \( R_c = 0.5 \) is used. Among 4QAM, 16QAM and 64QAM schemes, which is the most suitable scheme to employ in terms of bandwidth efficiency and transmission reliability, when we have an \( SNR \) of 6 dB?

(4 marks)

(d) The codeword \((y = [y_1 \ y_2 \ \ldots \ y_7]^T)\) of a rate-4/7 Hamming code can be generated based on its generator matrix \( G \) and information word \( x = [x_1 \ x_2 \ \ldots \ x_4]^T \) as follows:

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  y_4 \\
  y_5 \\
  y_6 \\
  y_7 \\
\end{bmatrix} = \begin{bmatrix}
  1 & 1 & 0 & 1 \\
  1 & 0 & 1 & 1 \\
  1 & 0 & 0 & 0 \\
  0 & 1 & 1 & 1 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix} \cdot \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
\end{bmatrix}
\]
The corresponding 3-bit syndrome \( s = [s_1 \ s_2 \ s_3]^T \) of this Hamming code can be computed as:

\[
\begin{bmatrix}
  s_1 \\
  s_2 \\
  s_3
\end{bmatrix}
= \begin{bmatrix}
  0 & 0 & 0 & 1 & 1 & 1 & 1 \\
  0 & 1 & 1 & 0 & 0 & 1 & 1 \\
  1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  \hat{y}_1 \\
  \hat{y}_2 \\
  \hat{y}_3 \\
  \hat{y}_4 \\
  \hat{y}_5 \\
  \hat{y}_6 \\
  \hat{y}_7
\end{bmatrix}
\]

where \( H \) is the parity check matrix and \( \hat{\mathbf{y}} = [\hat{y}_1 \ \hat{y}_2 \ldots \hat{y}_7]^T \) is the noise-corrupted received codeword. If \( \hat{\mathbf{y}} = [1100010] \). What are the corresponding syndrome \( s \) and the decoded 4-bit word \( \hat{\mathbf{x}} = [\hat{x}_1 \ \hat{x}_2 \ldots \hat{x}_4]^T \)?

(a) The output of a midrise uniform quantiser of input range \( \pm 5 \ V \) is coded using 4 bits/sample with the aid of a linear Pulse Coded Modulation (PCM).

(i) [2 marks] The maximum quantisation error \( e_{\text{max}} \):
The step size is \( q = 2(5)/2^4 = 0.625 \ V. \) Hence, \( e_{\text{max}} \) is given by: \( e_{\text{max}} = q/2 = 0.3125 \ V. \)

(ii) [3 marks] the quantisation error associated with an input sample of value \( s = -2.4 \ V. \)
Counting from interval \( l = 0 \) to \( l = 15 \) between -5V and +5V, the input sample \( s = -2.4 \ V \) falls in the \( l^{th} \) interval given by: \( l = \lfloor (s - (-5))/q \rfloor = \lfloor 4.16 \rfloor = 4. \)
The quantised output is given by the midpoint of the \( l^{th} \) interval:
\( \hat{s} = -5 + lq = -5 + q(4.16) = -2.1875 \ V. \) The quantisation error associated with an input sample of value \( s = -2.4 \ V \) is given by:
\( e = |s - \hat{s}| = 0.2125 \ V. \)

(iii) [3 marks] the uniformly distributed quantisation noise power \( \sigma_e^2 \) is given by:
\[
\sigma_e^2 = \int_{-q/2}^{q/2} e^2 \cdot p(e) de = \frac{1}{q} \int_{-q/2}^{q/2} e^2 de = \frac{1}{q} \left[ \frac{1}{3} e^3 \right]_{-q/2}^{q/2} = \frac{q^2}{12}
\]
\( = 0.032552 \)
(b)  
(i) [2 mark] The CDF is obtained integrating the PDF:

\[ P(u) = \int_{-\frac{1}{2}}^{u} 1 \, dx = u + \frac{1}{2} \]

for \(-\frac{1}{2} \leq u \leq \frac{1}{2} \).

(ii) [2 marks] The mean of \( x[n] \) is obtained from the mean of \( x \), and equals

\[ \mu_x = \int_{-\frac{1}{2}}^{\frac{1}{2}} x \, p(x) \, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} x \, \frac{1}{2} \, dx = \left. \frac{x^2}{4} \right|_{-\frac{1}{2}}^{\frac{1}{2}} = 0 \]

(iii) [4 marks] Consider that \( \sum_{k=-\infty}^{\infty} h[n] = 1 + 1 + 1 + 1 = 4 \). Conclude that the mean of the output is

\[ \mathcal{E}\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k] \mathcal{E}\{u[n-k]\} = 4 \cdot 0 = 0 \]

(c) [4 marks] The bandwidth efficiency is given by \( \eta = \log_2(M) \cdot R_c \), which is 1 BPS, 2 BPS and 3 BPS for 4QAM, 16QAM and 64QAM, respectively, when employing a half-rate channel code \( (R_c = 0.5) \).

For QAM modulation, the passband bandwidth is given by \( B = R_s \) (assuming ideal LPF). We want \( \eta \leq C/B \) for a potentially error-free transmission. We need to find out the maximum bandwidth efficiency value when the SNR is 6 dB (3.981).

\[ \eta \leq \log_2(1 + \text{SNR}) \]

\[ \eta \leq \log_2(1 + 3.981) = 2.316 \, \text{[BPS]} \]

Hence this SNR can support the half-rate coded 4QAM (1 BPS) and 16QAM (2 BPS) transmissions, but 16QAM is more suitable because it gives a higher bandwidth efficiency compared to 4QAM.

(d) [5 marks]

The related decoding information is given below.

\[
\begin{array}{ccc}
\hat{y} &=& y \oplus e \\
\hat{x} &=& s \\
110010 &=& 101 \\
0110 &=& 0110
\end{array}
\]

Syndrome = \( [101] \)
Corrected received sequence = \( [1100110] \)
The decoded 4-bit word is from the 3rd, 5th, 6th, 7th bits of the corrected sequence = \( [0110] \).

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