Sampling and Quantisation

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Learning Outcomes

By the end of this session you should be able to:

- Understand the effect of sampling on continuous-time signals in the time and frequency domains;

- Calculate a suitable sampling rate to avoid aliasing.
Why digitise analogue signals?

Digital Signal Processing
Why digitise analogue signals?

The digital computer can process discrete time signals using extremely flexible and powerful algorithms. However, most signals of interest are continuous time, which is how they almost always appear in nature.

In communications systems:

- With digital systems, it is easier to integrate different services such as video and voice, in the same transmission scheme.

- We can use several advanced coding and modulation techniques, resulting in improved communications performance.
Many physical systems operate in continuous time:
• Example: Leaky tank where the signal (water) is leaking.

Digital computations are done in discrete time:
• Example: MP3, CD, digital camera, everything on the web.
Sampling

Conversion of a continuous-time signal to a discrete-time signal.

Sampling allows the use of modern digital electronics to process, transmit, store and retrieve continuous-time signals.
**Sampling - Time Domain**

Consider a continuous-time signal $g(t)$, shown in the blue curve in the figure below. Suppose we sample $g(t)$ at a uniform rate, once every $T_s$ seconds. Hence, we obtain an infinite sequence of samples spaced at $T_s$ seconds apart and we denote this as $g(nT_s)$. $T_s$ is the sampling period and $f_S = \frac{1}{T_s}$ is the sampling rate.

Let $g_\delta(t)$ be the sampled version of $g(t)$, which can be represented as

$$g_\delta(t) = \sum_{n=\infty}^{+\infty} g(nT_s)\delta(t - nT_s).$$

$g_\delta(t)$ is shown by the red samples below.
Sampling - Time Domain

How does sampling affect the information contained in the signal?
We would like to sample in a way that preserves information. Hence, the sampling frequency should be selected properly.
Sampling-Frequency Domain: Revision of Fourier transform

When analysing time-domain signals, the frequency-domain representation is often very useful. The frequency-domain representation is the Fourier transform of the signal.

Consider a non-periodic deterministic signal $g(t)$, expressed as a function of $t$. Then the Fourier Transform of the signal $g(t)$ is given by

$$G(f) = \int_{-\infty}^{+\infty} g(t) \exp(-j2\pi ft) dt.$$
Sampling-Frequency Domain: Revision of Fourier transform

Consider a sinusoid signal with frequency 1Hz: $g(t) = \sin(2\pi t)$
Sampling-Frequency Domain: Revision of Fourier transform

Consider a periodic signal with period $T_0$, then the Fourier transform of this signal is discrete, i.e. has the effect of sampling the spectrum of the signal in the frequency domain.

Similarly, using the duality property of the Fourier transform, sampling the signal in the time-domain has the effect of making the spectrum of the signal periodic in the frequency domain.

The sampling process is usually described in the time-domain. Using sampling, an analogue signal is converted to a corresponding sequence of samples that are usually spaced uniformly in time.
The effects of discretisation in the time & frequency domains

\[ x(t) \quad X(j\omega) \]
\[ t(s) \quad \omega(rad/s) \]

\[ x(t) \quad x[n] \]
\[ t(s) \quad \omega(rad/s) \]
\[ \Omega = \omega T_s \]

\[ x(t) \quad x[n] \]
\[ t(s) \quad \omega(rad/s) \]
\[ \Omega_0 = 2\pi/N \]

\[ x(t) \quad x[n] \]
\[ t(s) \quad \omega(rad/s) \]
\[ T_0 = 2\pi/\omega_0 \]
\[ N \text{ samples} \]
Sampling-Frequency Domain

• According to the Discrete Fourier Transform – Discretisation in Time, we have:
  \[ x_s(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s); \]

• According to the Fourier series theorem and the duality property of the Fourier transform, the frequency domain of the sampled signal \( x_s(t) \) has been periodised, w.r.t. \( \omega_s = 2\pi/T_s \):
  \[
  X_s(j\omega) = \sum_{n=-\infty}^{\infty} X(j(\omega - n\omega_s))
  \]  
  (1)

• Duality of Fourier Transform: periodic waveform \( \circ \)——\( \bullet \) discrete spectrum
  discrete waveform \( \circ \)——\( \bullet \) periodic spectrum.

• Hence

\[ |X(j\omega)|, |X_s(j\omega)| \]
Sampling and Aliasing

How does sampling affect the information contained in the signal?
We would like to sample in a way that preserves information.

Information may be lost.
Nyquist Rate and Aliasing

- Nyquist sampling theorem states that all information of an analogue signal is retained if spectral repetitions do not overlap.

![Diagram showing Nyquist sampling theorem](image)

- Theoretically, the analogue signal can be perfectly reconstructed from the sampled version provided that a sampling frequency of at least twice the maximum bandwidth of the analogue signal is used:

\[
\omega_s = \frac{2 \times \pi}{T} > \omega_N = 2 \cdot \omega_{\text{max}} \quad \text{with Nyquist rate } \omega_N
\]  

(2)
Revision

1. Why do we digitise analogue signals?

2. What is Aliasing?

3. What is the minimum sampling frequency for sampling without loss of information? Explain!
Summary

• Effects of sampling are easy to visualize with Fourier representations.

• Signals that are bandlimited in frequency (e.g., $-W < \omega < W$) can be sampled without loss of information.

• The minimum sampling frequency for sampling without loss of information is called the Nyquist rate. The Nyquist rate is twice the highest frequency contained in a band-limited signal.

• Sampling at frequencies below the Nyquist rate causes aliasing. Aliasing can be eliminated by pre-filtering to remove frequency components that would otherwise alias.
Learning Outcomes

By the end of this session, you should be able to:

• Quantise a sampled signal;

• Characterise a digital-to-analogue converter;

• characterise an analogue-to-digital converter.
Revision

During the last session, we studied:

- Sampling in time and frequency domains;
- The minimum sampling rate for sampling without loss of information.

Any Questions?
Quantisation

We measure discrete amplitude in bits.

Real systems have limited resource in terms of the number of available bits for representing discrete signals.

Example: how many bits are needed to represent numbers in the range 1:20000

<table>
<thead>
<tr>
<th>Bits</th>
<th>Range</th>
<th>Bits</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>10</td>
<td>1024</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>11</td>
<td>2048</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>12</td>
<td>4096</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>13</td>
<td>8192</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>14</td>
<td>16384</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>15</td>
<td>32768</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>256</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>512</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Quantisation

24-bit quantised image  

1-bit quantised image
Quantisation: Linear Pulse Code Modulation

In linear PCM, the quantiser approximates a straight line with $2^R$ levels; example of linear PCM with $R = 3$ bits and asymmetric characteristic.

*midrise* uniform quantiser

*midtread* uniform quantiser
Quantisation: Linear Pulse Code Modulation

- In linear PCM, the quantiser approximates a straight line with $2^R$ levels; example of midrise linear PCM with $R = 3$ bits and asymmetric characteristic →

- the input signal has to use the range $[-V; +V]$ as best as possible, otherwise clipping or coarse resolution result; ($q = 2V/2^R$)

- quantisation noise ($e$): maximum $q/2$, uniformly distributed, additive;

- quantisation noise power:

\[
\sigma_e^2 = \int_{-q/2}^{q/2} e^2 \cdot p(e) \, de = \frac{1}{q} \int_{-q/2}^{q/2} e^2 \, de
\]

\[
= \frac{1}{q} \left[ \frac{1}{3} e^3 \right]_{-q/2}^{q/2} = \frac{(q/2)^2}{3} = \frac{q^2}{12} \quad (3)
\]
Signal-to-Quantisation Noise Ratio

- Consider a uniformly distributed input signal $x(t)$:

  \[ p(x) \]
  \[ -V \quad \frac{1}{2V} \quad \frac{1}{2V} \quad +V \]

- therefore, the signal power is

  \[
  \sigma_x^2 = \int_{-V}^{V} x^2 \cdot p(x) dx = \frac{1}{2V} \int_{-V}^{V} x^2 dx = \frac{1}{2V} \left[ \frac{1}{3} x^3 \right]_{-V}^{V} = \frac{V^2}{3} \quad (4)
  \]

- with (3) and $q = 2V/2^R$, the signal-to-quantisation noise ratio (SQNR) is given by

  \[
  \text{SQNR}_{dB} = 10 \cdot \log_{10} \left( \frac{V^2/3}{(2V/2^R)^2/12} \right) = 10 \cdot \log_{10} 2^{2R} \approx 6.02 \cdot R \quad \text{[dB]} \quad (5)
  \]

- Approximately 6 dB gain in SQNR for each bit added to a sample!
Dynamic Range

- Consider a uniformly distributed input signal $x(t)$:

- The Dynamic Range of a quantiser is defined as:

\[
R_d = \frac{\text{largest amplitude of the signal, which avoids clipping}}{\text{smallest amplitude of the signal, below which the signal variations go undetected}}
\]

\[
= \frac{V}{q/2}
\]

\[
= \frac{V}{(2V/2^R)/2} = 2^R
\]

\[
R_d(\text{dB}) = 20 \log_{10} 2^R = 6.02 \cdot R = \text{SQNR}_{\text{dB}}
\]

- Dynamic range = 72 dB requires $R \approx 72/6 = 12$ bits.
Analogue-to-Digital and Digital-to-Analogue Conversion

- For transmitting data, we will need to convert between analogue and digital signals;
- this is accomplished by analogue-to-digital and digital-to-analogue converters (ADCs / DACs);
- analogue signals, such as a speech signal, may require a conversion into the digital domain for applying sophisticated source coding algorithms:

![Analogue-to-Digital and Digital-to-Analogue Conversion Diagram]

- digital signals need to be converted to analogue quantities for transmission over an antenna link:
Analogue-to-Digital Conversion

- An ADC requires conditioning of the analogue signal, sampling and quantisation:

\[ x(t) \rightarrow \text{AAF} \rightarrow \text{signal conditioning} \rightarrow x[n] \rightarrow \text{Q} \rightarrow x_b[m] \]

- **sampling**: the selection of the sampling period \( T_s \) will affect the possible representation of the frequency content of \( x(t) \) by \( x[n] \);

- **quantisation**: the word length \( R \) will influence resolution, or — as derived in slide 23 — the signal-to-quantisation noise ratio (SQNR);

- **signal conditioning**: contains an anti-alias filter (AAF) and a gain in order to adjust the amplitude of \( x(t) \) to the input range of the quantiser \( Q \).
Sampling - Revision

- Effect of discretisation in time and frequency domain

- Nyquist Rate

\[ X(j\omega), |X_s(j\omega)| \]

\[ \omega = 0, \omega_{max}, \omega_s, 2\omega_s \]

Aliasing

No aliasing

\[ |X(j\omega)|, |X_s(j\omega)| \]
Signal Conditioning

- For a perfect signal representation, we have to sample a signal at least at the Nyquist rate;

- in order to avoid aliasing, we must fulfil: \( \omega_{\text{max}} < \omega_s/2 \);

- if the signal's content is not exactly known, we guarantee this by conditioning the input signal to the ADC by an anti-alias filter:

 This filter removes all frequency components above \( \omega_s/2 \), which would alias in the sampling stage.
Digital-to-Analogue Conversion

- For the conversion from the digital to the analogue domain, a DAC comprises of:

\[
\begin{align*}
&x_b[m] \
\rightarrow & \quad x_b[m] \quad \rightarrow \quad Q^{-1} \quad \rightarrow \quad x[n] \quad \rightarrow \quad x[nT_s] \quad \rightarrow \quad RCF \quad \rightarrow \quad x(t) \quad \rightarrow \quad x(t), x[nT_s] \\
&\text{Reconstruction Filter (RCF)}
\end{align*}
\]

- De-Quantisation: 
\[
x_b[m] = 111, 110, 101, 100, 000, 001, 010, 001
\]

\[
\begin{align*}
x[n] &= Q^{-1}(x_b[m]) \\
x[n] &= -\frac{7q}{2}, -\frac{5q}{2}, -\frac{3q}{2}, -\frac{q}{2}, \frac{q}{2}, \frac{3q}{2}, \frac{5q}{2}, \frac{7q}{2}
\end{align*}
\]

- The voltage signal at the output of \(Q^{-1}\) contains high frequency components, which need to be removed:

\[
\begin{align*}
\text{spectral repetitions} & \quad |X(j\omega)|, |X_s(j\omega)| \\
\text{desired spectrum} & \quad -2\omega_s, -\omega_s, 0, \omega_s, 2\omega_s
\end{align*}
\]
Reconstruction Filter

- In order to reconstruct the analogue signal $x(t)$, a lowpass filter is required to remove undesired high frequency components;

- this lowpass filter interpolates between adjacent sample values and is therefore termed reconstruction filter; requirement: stopband edge at $\omega_s/2$:

$$|H(j\omega)|$$

- with a perfect anti-alias filter and reconstruction filter, the concatenation of ADC and DAC imposes no loss on the signal (Nyquist);

- loss occurs due to (i) non-ideal anti-alias and reconstruction filters and (ii) quantisation.
Discretisation and Reconstruction Filter

Over-sampling: if \( x(t) \) is known.
Up-sampling: using interpolation if \( x(t) \) is not known.
Q1 The output of a midrise uniform quantiser of input range $\pm 5\, \text{V}$ is coded using $k = 3$ bits/sample with the aid of a linear Pulse Coded Modulation (PCM). Determine the quantisation error associated with an input sample of value $-1.3\, \text{V}$.

A1 The step size is $q = 2(5)/2^3 = 1.25\, \text{V}$.

The quantisation error associated with an input sample of value $s = -1.3\, \text{V}$:
Counting from interval $l = 0$ to $l = 7$ between $-5\, \text{V}$ and $+5\, \text{V}$, the input sample $s = -1.3\, \text{V}$ falls in the $l^{th}$ interval given by: $l = \lfloor (s - (-5))/q \rfloor = \lfloor 2.96 \rfloor = 2$.
The quantised output is given by the midpoint of the $l^{th}$ interval: $\hat{s} = -5 + lq + q/2 = -5 + q(l+0.5) = -1.875\, \text{V}$. The quantisation error associated with an input sample of value $s = -1.3\, \text{V}$ is given by: $e = |s - \hat{s}| = 0.575\, \text{V}$.

Q2 Assuming that:
1) the PCM scheme’s input signal does not exceed the quantizer’s dynamic range and that the signal is uniformly distributed across its entire dynamic range of $\pm V$;
2) the quantisation error \( e(t) \) imposed by the \( b \)-bit quantiser is uniformly distributed in the interval \([-q/2, q/2]\).

Q2a Under the above assumptions, determine the number of PCM bits required for achieving a dynamic range of 96dB in a linear PCM-based Compact Disc (CD) player.

A2a Dynamic range = 6.05 * R = 96, then R = 15.8. Therefore, 16 bits are required.

Q2b Given that the above-mentioned CD player uses a sampling frequency of 44.1 kHz, comment on the maximum bandwidth of the anti-aliasing low-pass filter and quantify the bit-rate required for the transmission of uncompressed CD-quality PCM-encoded music.

A2b Sampling frequency \( f_s = 44.1 \text{ KHz} \), then \( f_{max} \leq \frac{f_s}{2} = 22.05 \text{ KHz} \) Hence, AAF maximum frequency is around 22 KHz so a filter bandwidth of 20 KHz is adequate.

At sampling rate of 44.1 KHz the bitrate required for a mono channel is \( 44.1 \times 16 \) Kbit/s.
Tutorial: Sampling and Quantisation (2)

Q3 Describe the operation of an Analogue-to-Digital Convertor (ADC) using a diagram.

Q4 Describe the operation of a Digital-to-Analogue Convertor (DAC) using a diagram.

Q5 Why do we need to do over-sampling or up-sampling in a DAC?