1 Questions

1. A certain random variable has the probability density function of the form \( f_X(x) = \begin{cases} \frac{ce^{-4x}}{3} & x \geq 2 \\ 0 & \text{otherwise} \end{cases} \)
   
   (a) Find the constant \( c \).
   
   (b) Find the probability \( Pr(X > 4) \).

2. Let \( x[n] = v[n] + \frac{1}{2}v[n - 1] \), where \( v[n] \) is independently distributed, with mean \( \mu = 0 \) and variance \( \sigma^2 = 1 \). Is this signal WSS?

3. Consider a random variable \( x[\cdot] \) with a mean value of \(-8\) and variance \(1/6\).

   (a) Define the random variable \( z[\cdot] := -3x[\cdot] \); what is the numerical value of the expected value of \( z[\cdot] \)?
   
   (b) Define the random variable \( z[\cdot] := -3x[\cdot] \); what is the numerical value of the variance of \( z[\cdot] \)?
   
   (c) The sequence \( x[\cdot] \) is given as input to a linear, time-invariant system with impulse response
      \[
      h[k] := \begin{cases} \frac{1}{2}^k & \text{for } k \geq 0 \\ 0 & \text{otherwise} \end{cases},
      \]
      resulting in the output \( y[\cdot] \). What is the mean of \( y[\cdot] \)?
4. The output of a midrise uniform quantiser of input range ±5 V is coded using $k = 3$ bits/sample with the aid of a linear Pulse Coded Modulation (PCM). Determine the quantisation error associated with an input sample of value -1.3 V.

5. Assuming that:
   1) the PCM scheme’s input signal does not exceed the quantizer’s dynamic range and that the signal is uniformly distributed across its entire dynamic range of ±V;
   2) the quantisation error $e(t)$ imposed by the $b$-bit quantiser is uniformly distributed in the interval $[-q/2, q/2]$.

   a. Under the above assumptions, determine the number of PCM bits required for achieving a dynamic range of 96dB in a linear PCM-based Compact Disc (CD) player.

   b. Given that the above-mentioned CD player uses a sampling frequency of 44.1 kHz, comment on the maximum bandwidth of the anti-aliasing low-pass filter and quantify the bit-rate required for the transmission of uncompressed CD-quality PCM-encoded music.

6. How do you avoid aliasing in an Analogue-to-Digital Convertor (ADC), if the signal’s frequency content is not known?

7. Draw the block diagram of an Analogue-to-Digital Convertor (ADC) and explain the operations of signal conditioning, sampling and quantisation.
2 Answers

1. (a) Since

\[\int_{-\infty}^{+\infty} f_X(x) dx = 1, \text{ then} \]

\[c \times \int_{2}^{+\infty} e^{-4x} dx = 1 \]

\[c \times e^{-4x} \bigg |_{2}^{+\infty} = 1. \]

Then \(c \times \frac{e^{-8}}{2} = 1\) and finally \(c = 2e^8\).

(b) \(Pr(X > 4) = \int_{4}^{+\infty} f_X(x) dx = \int_{4}^{+\infty} 2e^{8}e^{-4x} dx = 2e^8(-e^{-4x}/(-2)) \bigg |_{4}^{+\infty} = e^{-8}\)

2. The mean of this signal is constant. Now

\[r_{xx}(n, m) = E \left\{ (v[n] + \frac{1}{2}v[n-1])(v[m] + \frac{1}{2}v[m-1]) \right\} \]

\[= E\{v[n]v[m]\} + \frac{1}{2}E\{v[n-1]v[m]\} + \frac{1}{2}E\{v[n]v[m-1]\} \]

\[+ \frac{1}{4}E\{v[n-1]v[m-1]\} \]

Apply the independence of the samples, and conclude that the second term equals \(\frac{1}{2}\) for \(n-1 = m\) and zero otherwise; the third one equals \(\frac{1}{2}\) for \(n = m - 1\) and zero otherwise; and the sum of the first and of the last one equals \(\frac{5}{4}E\{v[n]^2\} = \frac{5}{4}E\{(v[n] - \mu)^2\} = \frac{5}{4}\sigma^2 = \frac{5}{4}\) for \(n = m\), and zero otherwise.

Conclude that \(r_{xx}(n, m) = \frac{1}{2}\delta(n-1-m) + \frac{1}{2}\delta(n-m+1) + \frac{5}{4}\delta(n-m)\), which depends only on \(n-m\). Here \(\delta(k) = 0\) for \(k \neq 0\), \(\delta(0) = 1\). The signal is WSS.
3. (a) The mean of \( z \) is -3 times the mean of \( x \) i.e. 24.
(b) The variance of \( z \) is \( 9\text{var}(x) = \frac{3}{2} \).
(c) The mean of \( y \) is \( \mu_x \sum_{k=0}^{\infty} h[k] = -8 \frac{1}{1-\frac{1}{2}} = -16. \)

4. The step size is \( q = 2(5)/2^3 = 1.25 \text{ V}. \)
   The quantisation error associated with an input sample of value \( s = -1.3 \text{ V}: \)
   Counting from interval \( l = 0 \) to \( l = 7 \) between -5V and +5V, the input sample \( s = -1.3 \text{ V} \) falls in the \( l^{th} \) interval given by: \( l = \lfloor (s - (-5))/q \rfloor = \lfloor 2.96 \rfloor = 2. \)
The quantised output is given by the midpoint of the \( l^{th} \) interval:
   \( \hat{s} = -5 + lq + q/2 = -5 + q(l + 0.5) = -1.875 \text{ V}. \) The quantisation error associated with an input sample of value \( s = -1.3 \text{ V} \) is given by:
   \( e = |s - \hat{s}| = 0.575 \text{ V}. \)

5. (a) Dynamic range = 6.05 * R = 96, then \( R = 15.8. \) Therefore, 16 bits are required.
(b) sampling frequency \( f_s = 44.1 \text{ KHz}, \) then \( f_{max} \leq \frac{f_s}{2} = 22.05 \text{ KHz}. \)
   Hence, AAF maximum frequency is around 22 KHz so a filter bandwidth of 20 KHz is adequate.
   At sampling rate of 44.1 KHz the bit-rate required for a mono channel is \( 44.1 \times 16 \text{ Kbit/s}. \)

6. An anti-aliasing filter can be used in an ADC to remove all frequency components above half sampling frequency \( (f_s/2) \), in order to avoid aliasing.
7. The block diagram of an ADC is shown below, where \( x(t) \) is the analogue signal and \( x_{b}[m] \) is the digital signal:
• **signal conditioning**: contains an anti-alias filter (AAF) and a gain in order to adjust the amplitude of \( x(t) \) to the input range of the quantiser \( Q \).

• **sampling**: the selection of the sampling period \( T_s \) will affect the possible representation of the frequency content of \( x(t) \) by \( x[n] \);

• **quantisation**: the word length \( R \) will influence resolution, or the signal-to-quantisation noise ratio (SQNR).