Step by Step

Consider an initial value problem

\[ y'' + ay' + by = r(t), \quad y(0) = K_0, y'(0) = K_1 \]

where \( a \) and \( b \) are constant. Here \( r(t) \) is the given input (driving force) applied to the mechanical or electrical system and \( y(t) \) is the output (response to the input) to be obtained.
Step 1. Setting up the subsidiary equation

This is an algebraic equation for the transform

\[ Y = \mathcal{L}\{y\} \text{ obtained by transforming } \]

\[ [s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s) \]

where \( R(s) = \mathcal{L}\{r(t)\} \). Collecting the \( Y \)-terms, we have the subsidiary equation

\[ (s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s) \]
Step 2. Solution of the subsidiary equation

We divide by $s^2 + as + b$ and use the so-called transfer function

\[
Q(s) = \frac{1}{s^2 + as + b}
\]

to give the solution

\[
Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s)
\]
Step 3. Inversion of $Y(s)$ to obtain $y(t)$

We reduce $Y(s)$ (usually by partial fractions as in calculus) to a sum of terms whose inverses can be found from the tables, so that we obtain the solution

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
Example: Solve the initial value problem

\[ y'' - y = t, \quad y(0) = 1, \quad y'(0) = 1 \]

**Answer:**

\[ y(t) = e^t + \sinh t - t \]

\[ \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2} \]
Advantages of the Laplace Method:

- Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.
- Initial values are automatically taken care of.
- Complicated inputs $r(t)$ (right sides of linear ODEs) can be handled very efficiently, as we show in the next sections.
Example: Solve the initial value problem

\[ y'' + y' + 9y = 0, \quad y(0) = 0.16, y'(0) = 0 \]

**Answer:**

\[ y(t) = e^{-0.5t} (0.16\cos 2.96t + 0.027\sin 2.96t) \]
Summary of this lecture:

- Transforms of Derivatives & Integrals.
- Use Laplace transform to solve ODEs
Lecture 3

Unit Step Function (Heaviside Function).
Second Shifting Theorem (t-Shifting)
Summary of this lecture:

- Unit Step Function
- Second Shifting Theorem (\(t\)-Shifting)
Examples of inputs in engineering applications:

- Single waves
- Discontinuous driving force
- Short impulses
- ...  

**Unit Step Function** $u(t - \alpha)$ (this lecture)

**Dirac Delta Function** $\delta(t - \alpha)$ (next lecture)
Unit Step Function (Heaviside Function)

The unit step function (Heaviside function) \( u(t - a) \) is 0 for \( t < a \), has a jump of size 1 at \( t - a \) (where we can leave it undefined), and is 1 for \( t > a \), i.e.

\[
\begin{align*}
    u(t - a) &= \begin{cases} 
        0, & t < a \\
        1, & t > a 
    \end{cases} 
    \quad (a \geq 0)
\end{align*}
\]
Question 1: what is its Laplace transform?

Answer: \( \mathcal{L}\{u(t - a)\} = \frac{e^{-as}}{s} \)

Question 2: what is its use?

Let \( f(t) = 0 \) for \( t < 0 \). Then \( f(t - a)u(t - a) \) with \( a > 0 \) is \( f(t) \) **shifted (translated)** to the right by the amount \( a \).
Application of Unit Step Function: Examples

\[ f(t) = 5 \sin t \]
Application of Unit Step Function: Examples

\[ \begin{array}{c}
\text{Graph 1:}
\begin{array}{c}
k
\end{array}
\end{array} \]

\[ \begin{array}{c}
\text{Graph 2:}
\begin{array}{c}
4
\end{array}
\end{array} \]
Second Shifting Theorem ($t$–Shifting)

If a function $f(t)$ has the transform $F(s)$, then the ‘shifted function’

$$\tilde{f}(t) = f(t - a)u(t - a) = \begin{cases} 
0, & t < a \\
 f(t - a), & t > a 
\end{cases}$$

has the transform

$$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}F(s)$$
Example: Write the following function using unit step functions and calculate its Laplace transform

\[f(t) = \begin{cases} 
2, & 0 < t < 1 \\
\frac{1}{2}t^2, & 1 < t < \frac{\pi}{2} \\
\cos t, & t > \frac{\pi}{2}
\end{cases}\]

\[?\]
Example: Calculate the inverse transform of

\[ F(s) = \frac{e^{-s}}{s^2 + \pi^2} + \frac{e^{-2s}}{s^2 + \pi^2} + \frac{e^{-3s}}{(s + 2)^2} \]

**Answer:**

\[ f(t) = \frac{1}{\pi} \sin(\pi(t - 1))u(t - 1) \]

\[ + \frac{1}{\pi} \sin(\pi(t - 2))u(t - 2) \]

\[ + (t - 3) e^{-2(t-3)}u(t - 3) \]
Summary of this lecture:

- Unit Step Function
- Second Shifting Theorem (\(t\)-Shifting)
Summary of lectures till now:

- Basics of Laplace Transforms
  - Linearity
  - Transforms of Derivatives and Integrals
  - First (s-) and second (t-) shifting theorems
- Use Laplace Transforms to solve ODES
- Unit Step (Heaviside) Function
Find the current $i(t)$ in the RC-circuit in the following figure if a single rectangular wave with voltage $V_0$ is applied. The circuit is assumed to be quiescent before the wave is applied.
The Dynamics:

\[ Ri(t) + \frac{1}{C} \int_0^t i(t) \, dt = V_0 [u(t - a) - u(t - b)] \]

Solution:

\[ i(t) = \frac{V_0}{R} \left[ e^{\frac{-t-a}{RC}} u(t - a) - e^{\frac{-t-b}{RC}} u(t - b) \right] \]
Question

Find the response of the current $i(t)$ in the RLC-circuit in the following figure where

$$E(t) = 100\sin 400t \quad \text{if } 0 < t < 2\pi$$

and $E(t) = 0$ if $t > 2\pi$ and current and charge are initially zero.
The Dynamics:

\[ 0.1i' + 11i + 100 \int_0^t i(\tau) d\tau \]

\[ = (100\sin 400t)(1 - u(t - 2\pi)) \]

Solution:

?
Summary of lectures till now:

- Basics of Laplace Transforms
  - Linearity
  - Transforms of Derivatives and Integrals
  - First (s-) and second (t-) shifting theorems
- Use Laplace Transforms to solve ODES
- Unit Step (Heaviside) Function