MATH2047
Lectures 1~6
Laplace Transforms

Bing Chu
Electronics and Computer Science
University of Southampton
Email: b.chu@ecs.soton.ac.uk
Office: Building 1, Room 2007
Zhengzhou → Beijing → Sheffield → Oxford → Southampton
A little bit about me

2000~2007
Tsinghua Univ
Beijing, China
BSc, MSc

2007~2010
Sheffield Univ
PhD

2010~2012
Oxford Univ
A little bit about me

Learning control

Networked systems

Energy & sustainability
By the end of this week’s lectures, you should be able to

- use Laplace transforms to solve ODEs with constant coefficients and given boundary conditions.
Solving an ODE using the Laplace transform:

**Step 1.** The given ODE is transformed into an algebraic equation, called the subsidiary equation.

**Step 2.** The subsidiary equation is solved by purely algebraic manipulations.

**Step 3.** The solution in Step 2 is transformed back, resulting in the solution of the given problem.

*Fig. Solving an IVP by Laplace transforms*
The key motivation: the process of solving an ODE is simplified to an algebraic problem:

I. Problems are solved more directly: Initial value problems are solved without first determining a general solution. Nonhomogeneous ODEs are solved without first solving the homogeneous ODE.

II. More importantly, the use of the unit step function (Heaviside function) and Dirac’s delta make the method powerful for problems with inputs that have discontinuities or represent short impulses or complicated periodic functions.
Course overview

- Laplace transforms and ODEs: Basics (2 lectures)
- Unit step function and Dirac’s delta (2 lectures)
- Convolution theorem and more (2 lectures)
- Tutorial and Revision (2 lectures)
Course overview

- In class mini exercise
- Two revision/tutorial class (examples)
- Coursework assignments
- Final Examination
Course material

- Online slides
- Online notes (Dr Giles Richardson)
- Kreyzig E, Advanced Engineering Mathematics, Wiley. (Chapter 6 Laplace Transforms)
- Greenberg MD, Advanced Engineering Mathematics, CUP. (Chapter 5 on LT)
Lecture 1

Laplace Transform. Linearity. First Shifting Theorem ($s$-Shifting)
Summary of this lecture:

- Laplace Transform
- Linearity
- First Shifting Theorem (s-Shifting)
If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform is defined as

$$F(s) = \mathcal{L}\{f\} = \int_{0}^{\infty} e^{-st} f(t) \, dt$$

Furthermore, the given function $f(t)$ is called the inverse transform of $F(s)$ and is denoted by $\mathcal{L}^{-1}\{F\}$, we write

$$f(t) = \mathcal{L}^{-1}(F).$$
Example 1: Calculate the Laplace transform $F(s)$ of

$$f(t) = 1, \quad t \geq 0$$

Answer:

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt = \int_0^\infty e^{-st} \, dt$$

$$= -\frac{1}{s} e^{-st} \bigg|_0^\infty = \frac{1}{s} \quad (s > 0)$$
Example 2: Calculate the Laplace transform $F(s)$ of

$$f(t) = e^{at}, \quad t \geq 0$$

**Answer:**

$$F(s) = \int_{0}^{\infty} e^{-st} f(t) dt = \int_{0}^{\infty} e^{-st} e^{at} dt$$

$$= -\frac{1}{s-a} e^{-(s-a)t} \bigg|_{0}^{\infty} = \frac{1}{s-a}, \quad (s > a)$$
Laplace transform is a linear operation; that is, for any functions \( f(t) \) and \( g(t) \) whose transforms exist and any constants \( a \) and \( b \) the transform of

\[ af(t) + bg(t) \]

exists, and

\[ \mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}. \]
Two questions
Question 1: Calculate the Laplace transform $F(s)$ of

$$f(t) = \cos \omega t, \quad t \geq 0$$

**Answer:** $F(s) = \frac{s}{s^2 + \omega^2}$

Question 2: Calculate the Laplace transform $F(s)$ of

$$f(t) = \sin \omega t, \quad t \geq 0$$

**Answer:** $F(s) = \frac{\omega}{s^2 + \omega^2}$
### Basic Transforms

Some Functions $f(t)$ and Their Laplace Transforms

<table>
<thead>
<tr>
<th></th>
<th>$f(t)$</th>
<th>$\mathcal{L}(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>2</td>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>3</td>
<td>$t^2$</td>
<td>$\frac{2!}{s^3}$</td>
</tr>
<tr>
<td>4</td>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td></td>
<td>$(n = 0, 1, \ldots)$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$t^a$</td>
<td>$\frac{\Gamma(a+1)}{s^{a+1}}$</td>
</tr>
<tr>
<td></td>
<td>$(a$ positive)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
</tr>
<tr>
<td>7</td>
<td>$\cos \omega t$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>8</td>
<td>$\sin \omega t$</td>
<td>$\frac{\omega}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>9</td>
<td>$\cosh at$</td>
<td>$\frac{s}{s^2 - a^2}$</td>
</tr>
<tr>
<td>10</td>
<td>$\sinh at$</td>
<td>$\frac{a}{s^2 - a^2}$</td>
</tr>
<tr>
<td>11</td>
<td>$e^{at} \cos \omega t$</td>
<td>$\frac{s-a}{(s-a)^2 + \omega^2}$</td>
</tr>
<tr>
<td>12</td>
<td>$e^{at} \sin \omega t$</td>
<td>$\frac{\omega}{(s-a)^2 + \omega^2}$</td>
</tr>
</tbody>
</table>
First Shifting Theorem ($s$–Shifting)

If a function $f(t)$ has the transform $F(s)$, then $e^{at}f(t)$ has the transform $F(s - a)$ (where $s - a > k$). In formulas,

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$$

or, if we take the inverse on both sides,

$$e^{at}f(t) = \mathcal{L}^{-1}\{F(s - a)\}$$
Example 3: Calculate the Laplace transform $F(s)$ of

$$f(t) = e^{at} \cos \omega t, \quad t \geq 0$$

**Answer:**

$$F(s) = \frac{s-a}{(s-a)^2 + \omega^2}$$

**Question:** Calculate the Laplace transform $F(s)$ of

$$f(t) = e^{at} \sin \omega t, \quad t \geq 0$$
Question 3: find the inverse Laplace transform \( f(t) \) of

\[
F(s) = \frac{3s - 137}{s^2 + 2s + 401}
\]

Answer: \( f(t) = e^{-t}(3\cos(20t) - 7\sin(20t)) \)
Existence and Uniqueness of Laplace Transforms

- A function $f(t)$ has a Laplace transform if for all $t \geq 0$ and some constants $M$ and $k$ it satisfies the growth restriction $|f(t)| \leq Me^{kt}$.

- $f(t)$ need not be continuous, but it should not be too bad. The technical term (generally used in mathematics) is *piecewise continuity*. 
Existence Theorem for Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies the growth restriction condition for all $t \geq 0$ and some constants $M$ and $k$, then the Laplace transform $\mathcal{L}(f)$ exists for all $s > k$.

*Note:* if two continuous functions have the same transform, they are completely identical.
Summary of this lecture:

- Laplace Transform
- Linearity
- First Shifting Theorem (s-Shifting)
Lecture 2

Transforms of Derivatives and Integrals.

ODEs
Summary of this lecture:

- Transforms of Derivatives & Integrals
- Use Laplace transforms to solve ODEs
Laplace Transforms of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$

$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$

In this way, differentiation of the function corresponds to the multiplication of the transform by $s$. 
Transform of the Derivative $f^{(n)}$ of Any Order

Let $f, f', \ldots, f^{(n-1)}$ be continuous for all $t \geq 0$ and satisfy the growth restriction condition. Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Then the transform of $f^{(n)}$ satisfies

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$
Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geq 0$ and satisfies a growth restriction condition. Then

$$\mathcal{L} \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{F(s)}{s}, \quad \mathcal{L}^{-1} \left\{ \frac{F(s)}{s} \right\} = \int_0^t f(\tau) d\tau$$

In this way, integration of a function corresponds to the division of the transform by $s$. 
Question 1: Calculate the Laplace transform $F(s)$ of

$$f(t) = \cos \omega t, \quad t \geq 0$$

**Answer:** $F(s) = \frac{s}{s^2 + \omega^2}$

Question 2: Calculate the inverse transform $f(t)$ of

$$F(s) = \frac{1}{s(s^2 + \omega^2)}$$

**Answer:** $f(t) = \frac{1}{\omega^2} (1 - \cos \omega t)$
Step by Step

Consider an initial value problem

\[ y'' + ay' + by = r(t), \quad y(0) = K_0, y'(0) = K_1 \]

where \( a \) and \( b \) are constant. Here \( r(t) \) is the given input (driving force) applied to the mechanical or electrical system and \( y(t) \) is the output (response to the input) to be obtained.
Step 1. Setting up the subsidiary equation

This is an algebraic equation for the transform $Y = \mathcal{L}\{y\}$ obtained by transforming

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s) = \mathcal{L}\{r(t)\}$. Collecting the $Y$-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s)$$
Step 2. Solution of the subsidiary equation

We divide by \( s^2 + as + b \) and use the so-called transfer function

\[
Q(s) = \frac{1}{s^2 + as + b}
\]

to give the solution

\[
Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s)
\]
Step 3. Inversion of $Y(s)$ to obtain $y(t)$

We reduce $Y(s)$ (usually by partial fractions as in calculus) to a sum of terms whose inverses can be found from the tables, so that we obtain the solution

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
Example: Solve the initial value problem

\[ y'' - y = t, \quad y(0) = 1, \ y'(0) = 1 \]

**Answer:**

- **t-space**
  - Given problem:
    - \( y'' - y = t \)
    - \( y(0) = 1 \)
    - \( y'(0) = 1 \)

- **s-space**
  - Subsidiary equation:
    - \( (s^2 - 1)Y = s + 1 + 1/s^2 \)

- **Solution of given problem**
  - \( y(t) = e^t + \sinh t - t \)

- **Solution of subsidiary equation**
  - \( Y = \frac{1}{s - 1} + \frac{1}{s^2 - 1} - \frac{1}{s^2} \)
Advantages of the Laplace Method:

- Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.
- Initial values are automatically taken care of.
- Complicated inputs $r(t)$ (right sides of linear ODEs) can be handled very efficiently, as we show in the next sections.
Example: Solve the initial value problem

\[ y'' + y' + 9y = 0, \quad y(0) = 0.16, \ y'(0) = 0 \]

**Answer:**

\[ y(t) = e^{-0.5t} (0.16\cos 2.96t + 0.027\sin 2.96t) \]
Summary of this lecture:

- Transforms of Derivatives & Integrals.
- Use Laplace transform to solve ODEs
Lecture 3

Unit Step Function (Heaviside Function).
Second Shifting Theorem (t-Shifting)
Summary of this lecture:

- Unit Step Function
- Second Shifting Theorem (t-Shifting)
Examples of inputs in engineering applications:

- Single waves
- Discontinuous driving force
- Short impulses
- ...

**Unit Step Function** $u(t - a)$ *(this lecture)*

**Dirac Delta Function** $\delta(t - a)$ *(next lecture)*
The unit step function (Heaviside function) $u(t - a)$ is 0 for $t < a$, has a jump of size 1 at $t - a$ (where we can leave it undefined), and is 1 for $t > a$, i.e.

$$u(t - a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases} \quad (a \geq 0)$$
Question 1: what is its Laplace transform?

Answer: \[ \mathcal{L}\{u(t - a)\} = \frac{e^{-as}}{s} \]

Question 2: what is its use?

Let \( f(t) = 0 \) for \( t < 0 \). Then \( f(t - a)u(t - a) \) with \( a > 0 \) is \( f(t) \) **shifted (translated)** to the right by the amount \( a \).
Application of Unit Step Function: Examples

(A) \( f(t) = 5 \sin t \)
Application of Unit Step Function: Examples

\[ k \]

1 4 6 \[ t \]

\[ -k \]

0 2 4 6 8 10 \[ t \]
Second Shifting Theorem \((t-\text{Shifting})\)

If a function \(f(t)\) has the transform \(F(s)\), then the 'shifted function'

\[ f(t) = f(t - a)u(t - a) = \begin{cases} 
0, & t < a \\ 
f(t - a), & t > a 
\end{cases} \]

has the transform

\[
\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}F(s)
\]
Example: Write the following function using unit step functions and calculate its Laplace transform

\[ f(t) = \begin{cases} 
2, & 0 < t < 1 \\
\frac{1}{2} t^2, & 1 < t < \frac{\pi}{2} \\
\cos t, & t > \frac{\pi}{2}
\end{cases} \]

Answer: ?
Example: Calculate the inverse transform of

\[ F(s) = \frac{e^{-s}}{s^2 + \pi^2} + \frac{e^{-2s}}{s^2 + \pi^2} + \frac{e^{-3s}}{(s + 2)^2} \]

Answer:

\[ f(t) = \frac{1}{\pi} sin(\pi(t - 1))u(t - 1) \]
\[ + \frac{1}{\pi} sin(\pi(t - 2))u(t - 2) + (t - 3) e^{-2(t - 3)}u(t - 3) \]
Summary of this lecture:

- Unit Step Function
- Second Shifting Theorem ($t$-Shifting)
Summary of lectures till now:

- Basics of Laplace Transforms
  - Linearity
  - Transforms of Derivatives and Integrals
  - First (s-) and second (t-) shifting theorems
- Use Laplace Transforms to solve ODES
- Unit Step (Heaviside) Function
Find the current $i(t)$ in the RC-circuit in the following figure if a single rectangular wave with voltage $V_0$ is applied. The circuit is assumed to be quiescent before the wave is applied.
Answer

The Dynamics:

\[ Ri(t) + \frac{1}{C} \int_0^t i(t) dt = V_0 [u(t - a) - u(t - b)] \]

Solution:

\[ i(t) = \frac{V_0}{R} \left[ e^{-\frac{t-a}{RC}} u(t-a) - e^{-\frac{t-b}{RC}} u(t-b) \right] \]
Question

Find the response of the current $i(t)$ in the RLC-circuit in the following figure where

$$ E(t) = 100\sin 400t \quad \text{if } 0 < t < 2\pi $$

and $E(t) = 0$ if $t > 2\pi$ and current and charge are initially zero.
The Dynamics:

\[ 0.1 i' + 11i + 100 \int_0^t i(\tau) d\tau \]

\[ = (100 \sin 400t)(1 - u(t - 2\pi)) \]

Solution:
Summary of lectures till now:

- Basics of Laplace Transforms
  - Linearity
  - Transforms of Derivatives and Integrals
  - First (s-) and second (t-) shifting theorems
- Use Laplace Transforms to solve ODES
- Unit Step (Heaviside) Function
Lecture 4

Short Impulses. Dirac’s Delta Function. Partial Fractions
Summary of this lecture:

- Dirac’s Delta Function
- More on Partial Fractions
Examples of short impulse:

- Airplane making a ‘hard’ landing
- Mechanical system being hit by a hammer blow
- Ship being hit by a single high wave
- ...

*Dirac Delta Function $\delta(t - \alpha)$ (this lecture)*
To model situations of that type, use the function

\[ f_k(t-a) = \begin{cases} 
1/k & \text{if } a \leq t \leq a+k \\
0 & \text{otherwise}
\end{cases} \]
Dirac Delta Function

Let $k \to 0$, denote the limit as $\delta(t - a)$,

$$\delta(t - a) = \lim_{k \to 0} f_k(t - a)$$

$\delta(t - a)$ is called **Dirac Delta Function** or unit impulse function.

Note:

$$\delta(t - a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} \quad \int_0^\infty \delta(t - a)dt = 1,$$
Dirac Delta Function: Properties

**Sifting property:**

\[
\int_0^\infty g(t)\delta(t - a) \, dt = g(a)
\]

**Laplace Transform of** \(\delta(t - a)\):

\[
\mathcal{L}\{\delta(t - a)\} = e^{-as}
\]
Determine the Hammerblow Response of a Mass-Spring System which can be modelled using

\[ y'' + 3y' + 2y = \delta(t - 1), \quad y(0) = 0, y'(0) = 0 \]

**Answer:**

\[ y(t) = [e^{-(t-1)} - e^{-2(t-1)}]u(t - 1) \]
Step 3. Inversion of $Y(s)$ to obtain $y(t)$

We reduce $Y(s)$ (usually by partial fractions as in calculus) to a sum of terms whose inverses can be found from the tables, so that we obtain the solution

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{G(s)}\right\}$$
More on Partial Fractions

An unrepeated factors $s - a$ in $G(s)$ requires a single partial fraction

\[ \frac{A}{s - a} \]

The inverse:

\[ Ae^{at} \]
More on Partial Fractions

Repeated real factors \((s - a)^2, (s - a)^3\), etc. in \(G(s)\) requires partial fractions

\[
\frac{A_1}{s - a} + \frac{A_2}{(s - a)^2} + \frac{A_3}{(s - a)^3} + \cdots
\]

The inverse:

\[
(A_1 + A_2 t + \frac{1}{2} A_3 t^2 + \cdots) e^{at}
\]
More on Partial Fractions

Unrepeated complex factors \((s - \alpha)(s - \bar{\alpha})\) in \(G(s)\) requires a partial fraction

\[
\frac{As + B}{(s - \text{Re}(\alpha))^2 + \text{Im}(\alpha)^2}
\]

The inverse:

?

Repeated complex factors \([ (s - \alpha)(s - \bar{\alpha}) ]^2\)?

Next lecture!
Example: Solve the initial value problem

\[ y'' + 2y' + 2y = r(t), \]

where \( r(t) = 10\sin2t \) if \( 0 < t < \pi \) and \( 0 \) if \( t > \pi \);

\[ y(0) = 0, \ y'(0) = -5 \]

Answer:

\[ y(t) = 3e^{-t}\cos t - 2\cos2t - \sin2t, \quad 0 < t < \pi \]

\[ y(t) = e^{-t}[(3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t], \quad t > \pi \]
Summary of this lecture:

- Dirac’s Delta Function
- More on Partial Fractions
Lecture 5

Convolution
Summary of this lecture:

- Convolution theorem
- Use convolution to solve nonhomogeneous ODEs
We know

\[ \mathcal{L}^{-1}\{F(S) + G(s)\} = f(t) + g(t) \]

**Question:** is the following true or not

\[ \mathcal{L}^{-1}\{F(s)G(s)\} = f(t)g(t) \]

**Answer:** No!

**Example:** \( f(t) = e^{at} \) and \( g(t) = 1 \)
Convolution Theorem

If two functions $f(t)$ and $g(t)$ satisfies the conditions of the existence theorem and

$$\mathcal{L}\{f(t)\} = F(s), \mathcal{L}\{g(t)\} = G(s),$$

then $F(s)G(s)$ is the transform of the convolution of $f(t)$ and $g(t)$ defined as follows

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t) = \int_0^t f(\tau)g(t - \tau)\,d\tau$$
We know

$$\mathcal{L}^{-1}\{F(S) + G(s)\} = f(t) + g(t)$$

**Question:** is the following true or not

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t)g(t)$$

**Answer:** No!

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau$$

**Example:** $f(t) = e^{at}$ and $g(t) = 1$
Example: Calculate the inverse transform of

$$F(s) = \frac{1}{(s^2 + \omega^2)^2}$$

**Answer:**

$$f(t) = \frac{1}{2\omega^2} \left[ -tcos\omega t + \frac{sin\omega t}{\omega} \right]$$
Consider the following ODE

\[ y'' + ay' + by = r(t) \]

Solution

\[ Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s) \]

Using the convolution theorem

\[ \mathcal{L}^{-1}\{R(s)Q(s)\} = \int_0^t q(t - \tau)r(\tau)d\tau \]
Use convolution to solve the initial value problem

\[ y'' + 3y' + 2y = r(t), \]

where \( r(t) = 1 \) if \( 1 < t < 2 \) and 0 otherwise;

\[ y(0) = 0, y'(0) = 0 \]

**Answer:**

\[ y(t) = f(t - 1)u(t - 1) - f(t - 2)u(t - 2) \]

\[ f(t) = \frac{1}{2} e^{-t} - \frac{1}{2} e^{-2t} \]
A Volterra Integral Equation of the Second Kind:

\[ y(t) - \int_0^t y(\tau) \sin(t - \tau) d\tau = t \]

**Solution:**

\[ Y(s) = \frac{s^2 + 1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \Rightarrow y(t) = t + \frac{t^3}{6} \]
Summary of this lecture:

- Convolution theorem
- Use convolution to solve nonhomogeneous ODEs
Lecture 6

Differentiation and Integration of Transforms. Change of scale. Periodic functions
Summary of this lecture:

- Differentiation and Integration of Transforms
- Change of scale
- Laplace transform of periodic functions
Differentiation of Transforms

If a function $f(t)$ satisfies the conditions of the existence theorem and

$$F(s) = \mathcal{L}\{f(t)\}$$

then

$$\mathcal{L}\{tf(t)\} = -F'(s), \quad \mathcal{L}^{-1}\{F'(s)\} = -tf(t)$$

In this way, differentiation of the transform of a function corresponds to the multiplication of the function by $-t$. 
Integration of Transforms

If a function $f(t)$ satisfies the conditions of the existence theorem and the limit of $f(t)/t$ as $t$ approaches 0 from the right, exists, then for $s > k$

$$
\mathcal{L} \left\{ \frac{f(t)}{t} \right\} = \int_s^\infty F(\tilde{s})d\tilde{s}, \quad \mathcal{L}^{-1} \left\{ \int_s^\infty F(\tilde{s})d\tilde{s} \right\} = \frac{f(t)}{t}
$$

In this way, integration of the transform of a function corresponds to the division of the function by $t$. 
### Example: Differentiation of Transforms

<table>
<thead>
<tr>
<th>$\mathcal{L}(f)$</th>
<th>$f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{(s^2 + \beta^2)^2}$</td>
<td>$\frac{1}{2\beta^3}(\sin \beta t - \beta t \cos \beta t)$</td>
</tr>
<tr>
<td>$\frac{s}{(s^2 + \beta^2)^2}$</td>
<td>$\frac{1}{2\beta}\sin \beta t$</td>
</tr>
<tr>
<td>$\frac{s^2}{(s^2 + \beta^2)^2}$</td>
<td>$\frac{1}{2\beta}(\sin \beta t + \beta t \cos \beta t)$</td>
</tr>
</tbody>
</table>
Example: Calculate the inverse transform

\[ F(s) = \ln \frac{s^2 + \omega^2}{s^2} \]

Answer:

\[ f(t) = \frac{2}{t} (1 - \cos \omega t) \]
Change of Scale

If a function $f(t)$ satisfies the conditions of the existence theorem and

$$F(s) = \mathcal{L}\{f(t)\}$$

then for $a > 0$

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right), \quad \mathcal{L}^{-1}\left\{\frac{1}{a} F\left(\frac{s}{a}\right)\right\} = f(at)$$
The Laplace transform of a periodic function $f(t)$ with period $T$ is

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) \, dt$$

**Question**: why is this useful?
Example: Calculate the Laplace transform of the saw-tooth wave

Answer:

\[ F(s) = \frac{k}{ps^2} - \frac{ke^{-ps}}{s(1 - e^{-ps})} \]
Final Value Theorem

If a function $f(t)$ has Laplace transform $F(s)$, then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Initial Value Theorem

If a function $f(t)$ has Laplace transform $F(s)$ and $f(t)$ does not contain $\delta(\cdot)$ (or its derivative) at zero, then

$$f(0) = \lim_{s \to \infty} sF(s)$$
Summary of this lecture:

- Differentiation and Integration of Transforms
- Change of scale
- Laplace transform of periodic functions
Summary of lectures since last tutorial:

- More properties of Laplace Transforms
  - Convolution theorem
  - Differentiation and Integration of Transforms
  - Change of scale
  - Periodic functions
- Dirac Delta function
- Use convolution to solve ODES
Question 1: Find the inverse transform

\[ F(s) = \frac{1}{(s + 1)(1 + s^2)} \]

**Answer:** \( f(t) = \frac{1}{2}e^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t \)

Question 2: Find the inverse transform

\[ F(s) = \frac{e^{-s}}{s(s + 1)} \]

**Answer:** \( f(t) = (1 - e^{-(t-1)})u(t - 1) \)
Question 3: use Laplace transform to solve

\[ y'' + 4y' + 5y = 0, \quad y(0) = 1, y'(0) = -3 \]

**Answer:** \( y(t) = e^{-2t}(\cos t - \sin t) \)

Question 4: use Laplace transform to solve

\[ y'' + 3y' + 2y = t + e^{-t}, \quad y(0) = 0, y'(0) = 0 \]

**Answer:** \( y(t) = \frac{1}{2}t - \frac{3}{4} + te^{-t} + \frac{3}{4}e^{-t} \)
Question 5: use Laplace transform to solve

\[ y'' + 16y = 8u(t - 4), \quad y(0) = 1, y'(0) = 0 \]

Answer: ?
Summary of lectures since last tutorial:

- More properties of Laplace Transforms
  - Convolution theorem
  - Differentiation and Integration of Transforms
  - Change of scale
  - Periodic functions
- Dirac Delta function
- Use convolution to solve ODES
MATH2047
Lectures 1~6
Laplace Transforms

Bing Chu
Electronics and Computer Science
University of Southampton
Email: b.chu@ecs.soton.ac.uk
Office: Building 1, Room 2007
MATH2047
Laplace Transforms
Key Properties and Formulas

Bing Chu
Electronics and Computer Science
University of Southampton
Email: b.chu@ecs.soton.ac.uk
Office: Building 1, Room 2007
Course material

- Online slides
- Online notes (Dr Giles Richardson)
- Kreyzig E, Advanced Engineering Mathematics, Wiley. (Chapter 6 Laplace Transforms)
- Greenberg MD, Advanced Engineering Mathematics, CUP. (Chapter 5 on LT)
# Laplace Transforms: Key Properties

<table>
<thead>
<tr>
<th>Formula</th>
<th>Name, Comments</th>
<th>Sec.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(s) = \mathcal{L}{f(t)} = \int_0^\infty e^{-st}f(t),dt$</td>
<td>Definition of Transform</td>
<td>6.1</td>
</tr>
<tr>
<td>$f(t) = \mathcal{L}^{-1}{F(s)}$</td>
<td>Inverse Transform</td>
<td>6.1</td>
</tr>
<tr>
<td>$\mathcal{L}{af(t) + bg(t)} = a\mathcal{L}{f(t)} + b\mathcal{L}{g(t)}$</td>
<td>Linearity</td>
<td>6.1</td>
</tr>
<tr>
<td>$\mathcal{L}{e^{at}f(t)} = F(s - a)$</td>
<td>$s$-Shifting (First Shifting Theorem)</td>
<td>6.1</td>
</tr>
<tr>
<td>$\mathcal{L}^{-1}{F(s - a)} = e^{at}f(t)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Note: Section numbers refer to Kreyszig E.'s book*
### Laplace Transforms: Key Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}(f')$</td>
<td>$s\mathcal{L}(f) - f(0)$</td>
<td>Differentiation of Function</td>
</tr>
<tr>
<td>$\mathcal{L}(f'')$</td>
<td>$s^2\mathcal{L}(f) - sf(0) - f'(0)$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}(f^{(n)})$</td>
<td>$s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0)$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}\left{ \int_0^t f(\tau) , d\tau \right}$</td>
<td>$\frac{1}{s} \mathcal{L}(f)$</td>
<td>Integration of Function</td>
</tr>
<tr>
<td>$(f * g)(t)$</td>
<td>$\int_0^t f(\tau)g(t - \tau) , d\tau$</td>
<td>Convolution</td>
</tr>
<tr>
<td></td>
<td>$= \int_0^t f(t - \tau)g(\tau) , d\tau$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}(f * g)$</td>
<td>$\mathcal{L}(f)\mathcal{L}(g)$</td>
<td></td>
</tr>
</tbody>
</table>

**References:**
- [6.2](#)
- [6.5](#)
### Laplace Transform: Key Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Equation</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}{f(t - a)u(t - a)} = e^{-as}F(s)$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\mathcal{L}^{-1}{e^{-as}F(s)} = f(t - a)u(t - a)$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\mathcal{L}{tf(t)} = -F'(s)$</td>
<td>Differentiation of Transform</td>
<td>6.6</td>
</tr>
<tr>
<td>$\mathcal{L}\left{\frac{f(t)}{t}\right} = \int_s^\infty F(\tilde{s})d\tilde{s}$</td>
<td>Integration of Transform</td>
<td>6.6</td>
</tr>
<tr>
<td>$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}}\int_0^p e^{-st}f(t),dt$</td>
<td>$f$ Periodic with Period $p$</td>
<td>6.4 Project 16</td>
</tr>
</tbody>
</table>
## Laplace Transforms: General Formulas

<table>
<thead>
<tr>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
<th>$f(t)$</th>
<th>Sec.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/s$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$1/s^2$</td>
<td>$t$</td>
<td></td>
</tr>
<tr>
<td>$1/s^n$ $(n = 1, 2, \cdots)$</td>
<td>$t^{n-1}/(n-1)!$</td>
<td>6.1</td>
</tr>
<tr>
<td>$1/\sqrt{s}$</td>
<td>$1/\sqrt{\pi t}$</td>
<td></td>
</tr>
<tr>
<td>$1/s^{3/2}$</td>
<td>$2\sqrt{t/\pi}$</td>
<td></td>
</tr>
<tr>
<td>$1/s^a$ $(a &gt; 0)$</td>
<td>$t^{a-1}/\Gamma(a)$</td>
<td></td>
</tr>
<tr>
<td>$1/(s-a)$</td>
<td>$e^{at}$</td>
<td></td>
</tr>
<tr>
<td>$1/(s-a)^2$</td>
<td>$te^{at}$</td>
<td></td>
</tr>
<tr>
<td>$1/(s-a)^n$ $(n = 1, 2, \cdots)$</td>
<td>$1/(n-1)!t^{n-1}e^{at}$</td>
<td>6.1</td>
</tr>
<tr>
<td>$1/(s-a)^k$ $(k &gt; 0)$</td>
<td>$1/\Gamma(k)t^{k-1}e^{at}$</td>
<td></td>
</tr>
</tbody>
</table>
### Laplace Transforms: General Formulas

<table>
<thead>
<tr>
<th>#</th>
<th>Formula</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>( \frac{1}{(s - a)(s - b)} ) ((a \neq b))</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>( \frac{s}{(s - a)(s - b)} ) ((a \neq b))</td>
<td>( \frac{1}{a - b} )(e^{at} - e^{bt})</td>
</tr>
<tr>
<td>13</td>
<td>( \frac{1}{s^2 + \omega^2} )</td>
<td>( \frac{1}{\omega} ) sin ( \omega t )</td>
</tr>
<tr>
<td>14</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
<td>cos ( \omega t )</td>
</tr>
<tr>
<td>15</td>
<td>( \frac{1}{s^2 - a^2} )</td>
<td>( \frac{1}{a} ) sinh ( at )</td>
</tr>
<tr>
<td>16</td>
<td>( \frac{s}{s^2 - a^2} )</td>
<td>cosh ( at )</td>
</tr>
<tr>
<td>17</td>
<td>( \frac{1}{(s - a)^2 + \omega^2} )</td>
<td>( \frac{1}{\omega} ) e^{\omega t} sinh ( \omega t )</td>
</tr>
<tr>
<td>18</td>
<td>( \frac{s - a}{(s - a)^2 + \omega^2} )</td>
<td>e^{at} cos ( \omega t )</td>
</tr>
</tbody>
</table>

\[ 6.1 \]
### Laplace Transforms: General Formulas

<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
<th></th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>( \frac{1}{s(s^2 + \omega^2)} )</td>
<td></td>
<td>( \frac{1}{\omega^2} (1 - \cos \omega t) )</td>
</tr>
<tr>
<td>20</td>
<td>( \frac{1}{s^2(s^2 + \omega^2)} )</td>
<td></td>
<td>( \frac{1}{\omega^3} (\omega t - \sin \omega t) )</td>
</tr>
<tr>
<td>21</td>
<td>( \frac{1}{(s^2 + \omega^2)^2} )</td>
<td></td>
<td>( \frac{1}{2\omega^3} (\sin \omega t - \omega t \cos \omega t) )</td>
</tr>
<tr>
<td>22</td>
<td>( \frac{s}{(s^2 + \omega^2)^2} )</td>
<td></td>
<td>( \frac{t}{2\omega} \sin \omega t )</td>
</tr>
<tr>
<td>23</td>
<td>( \frac{s^2}{(s^2 + \omega^2)^2} )</td>
<td></td>
<td>( \frac{1}{2\omega} (\sin \omega t + \omega t \cos \omega t) )</td>
</tr>
<tr>
<td>24</td>
<td>( \frac{s}{(s^2 + a^2)(s^2 + b^2)} ) ( (a^2 \neq b^2) )</td>
<td></td>
<td>( \frac{1}{b^2 - a^2} (\cos at - \cos bt) )</td>
</tr>
</tbody>
</table>

---

**MATH2047: Mathematics for Electronics and Electrical Engineering**

Laplace Transforms

09/10/2014
<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>$\frac{1}{s^4 + 4k^4}$</td>
<td>$\frac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$</td>
</tr>
<tr>
<td>26</td>
<td>$\frac{s}{s^4 + 4k^4}$</td>
<td>$\frac{1}{2k^2} \sin kt \sinh kt$</td>
</tr>
<tr>
<td>27</td>
<td>$\frac{1}{s^4 - k^4}$</td>
<td>$\frac{1}{2k^3}(\sinh kt - \sin kt)$</td>
</tr>
<tr>
<td>28</td>
<td>$\frac{s}{s^4 - k^4}$</td>
<td>$\frac{1}{2k^2}(\cosh kt - \cos kt)$</td>
</tr>
<tr>
<td>29</td>
<td>$\sqrt{s - a} - \sqrt{s - b}$</td>
<td>$\frac{1}{2\sqrt{\pi}t^3}(e^{bt} - e^{at})$</td>
</tr>
<tr>
<td>30</td>
<td>$\frac{1}{\sqrt{s + a} \sqrt{s + b}}$</td>
<td>$e^{-(a+b)t/2}I_0\left(\frac{a - b}{2t}\right)$</td>
</tr>
<tr>
<td>31</td>
<td>$\frac{1}{\sqrt{s^2 + a^2}}$</td>
<td>$J_0(at)$</td>
</tr>
</tbody>
</table>
### Laplace Transforms: General Formulas

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{s}{(s - a)^{3/2}}$</td>
<td>$(k &gt; 0)$</td>
</tr>
<tr>
<td>$\frac{1}{(s^2 - a^2)^k}$</td>
<td>$\frac{1}{\sqrt{\pi t}} e^{at}(1 + 2at)$</td>
</tr>
<tr>
<td>$\frac{1}{\Gamma(k)} \left( \frac{t}{2a} \right)^{k-1/2} I_{k-1/2}(at)$</td>
<td>$I_{5.5}$</td>
</tr>
<tr>
<td>$e^{-as}/s$</td>
<td>$u(t - a)$</td>
</tr>
<tr>
<td>$e^{-as}$</td>
<td>$\delta(t - a)$</td>
</tr>
<tr>
<td>$\frac{1}{s} e^{-k/s}$</td>
<td>$J_0(2 \sqrt{kt})$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{s}} e^{-k/s}$</td>
<td>$\frac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$</td>
</tr>
<tr>
<td>$\frac{1}{s^{3/2}} e^{k/s}$</td>
<td>$\frac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$</td>
</tr>
<tr>
<td>$e^{-k\sqrt{s}}$</td>
<td>$\frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$</td>
</tr>
<tr>
<td>$(k &gt; 0)$</td>
<td></td>
</tr>
</tbody>
</table>

---

**MATH2047: Mathematics for Electronics and Electrical Engineering**

Laplace Transforms

09/10/2014
### Laplace Transforms: General Formulas

<table>
<thead>
<tr>
<th></th>
<th>( F(s) = \mathcal{L}{f(t)} )</th>
<th>( f(t) )</th>
<th>Sec.</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>( \frac{1}{s} \ln s )</td>
<td>(-\ln t - \gamma ) (( \gamma \approx 0.5772 ))</td>
<td>( \gamma ) 5.5</td>
</tr>
<tr>
<td>41</td>
<td>( \ln \frac{s - a}{s - b} )</td>
<td>( \frac{1}{t} (e^{bt} - e^{at}) )</td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>( \ln \frac{s^2 + \omega^2}{s^2} )</td>
<td>( \frac{2}{t} (1 - \cos \omega t) )</td>
<td>6.6</td>
</tr>
<tr>
<td>43</td>
<td>( \ln \frac{s^2 - a^2}{s^2} )</td>
<td>( \frac{2}{t} (1 - \cosh at) )</td>
<td></td>
</tr>
<tr>
<td>44</td>
<td>( \arctan \frac{\omega}{s} )</td>
<td>( \frac{1}{t} \sin \omega t )</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>( \frac{1}{s} \arccot s )</td>
<td>( \text{Si}(t) )</td>
<td>App. A3.1</td>
</tr>
</tbody>
</table>
MATH2047
Laplace Transforms
Key Properties and Formulas

Bing Chu
Electronics and Computer Science
University of Southampton
Email: b.chu@ecs.soton.ac.uk
Office: Building 1, Room 2007