The aim of the course is to introduce basic topics in \textit{mechanics}, \textit{fields} and \textit{waves} for use in subsequent courses on devices, electricity and magnetism and optoelectronics.
Part II (S2): Oscillations and Waves

- Simple Harmonic Motion
- Coupled Oscillators
- Electromagnetic Oscillations
- Travelling Waves and Wave Equation
- Standing Waves and Interference
- Electromagnetic Waves
Course overview

- In class mini exercise
- Tutorial and Revision class
- Final revision week
- Final Examination (50%)
- Participate! Participate! Participate!
- Office hour (Wens. 12:00 – 13:00)/Email
Tips on Revision

- Online slides
- Read the textbook
- Worked questions in the book
- Tutorial questions
- Voluntary coursework - Friday
Simple Harmonic Motion

- Hooke’s law: $F = -kx$
- Newton’s law: 
  $$F = ma = md^2x/dt^2$$
- Equation of SHM (a differential equation for $x(t)$)
  $$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$$
Solve the SHM equation

- **Solution** \( x(t) = x_m \cos(\omega t + \phi) \)
- **Angular frequency**
  \[ \omega = \sqrt{\frac{k}{m}} \]
- **Period and frequency**
  \[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \quad f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \]
- **An interesting observation:** all the above quantities only depend on \( m \) and \( k \)!
Energy in Simple Harmonic Oscillator

- Potential energy:

\[ U(t) = \frac{1}{2} kx^2 = \frac{1}{2} kx_m^2 \cos^2(\omega t + \phi) \]

- Kinetic energy:

\[ K(t) = \frac{1}{2} m\dot{x}^2 = \frac{1}{2} m\omega^2 x_m^2 \sin^2(\omega t + \phi) \]

\[ = \frac{1}{2} kx_m^2 \sin^2(\omega t + \phi) \]

- Total energy:

\[ E(t) = U(t) + K(t) = \frac{1}{2} kx_m^2 \]
Hooke’s law: \( F_r = -kx \)

Damping: \( F_d = -bv \) --- resistive force

Newton’s law: \( F = ma = md^2x/dt^2 \)

Equation of damped Harmonic motion

\[
m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0
\]
• Equation of damped harmonic motion

\[ m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0 \]

• Try: \( x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi) \)

• It works and

\[ \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \]

where \( x_m \) and \( \phi \) are determined by the initial conditions.
Solve Damped Harmonic Motion Equation

- \( x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi) \)

\[
\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}
\]

- \( b = 0 \Rightarrow \omega' = \omega = \sqrt{\frac{k}{m}} \)

- \( b \ll \sqrt{km} \Rightarrow \omega' \approx \omega \)

- Increasing \( b \)?
Energy in the Damped Harmonic Oscillator

- **Potential energy:**
  \[ U(t) = \frac{1}{2} kx^2 = \frac{1}{2} kx_m e^{-bt/m} \cos^2(\omega't + \phi) \]

- **Kinetic energy:**
  \[ K(t) = \frac{1}{2} m v^2 \approx \frac{1}{2} m \omega'^2 x_m^2 e^{-bt/m} \sin^2(\omega't + \phi) \]
  \[ \approx \frac{1}{2} kx_m^2 e^{-bt/m} \sin^2(\omega't + \phi) \]

- **Total energy:**
  \[ E(t) = U(t) + K(t) \approx \frac{1}{2} kx_m^2 e^{-bt/m} \]
Forced Oscillation

- Hooke’s law: \( F = -kx \)  
  Damping force: \( F_d(t) = -bv \)
- Driving force: \( F_e = F_0 \cos(\omega_d t) \)
- Equation of forced harmonic motion
  \[
  m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos(\omega_d t)
  \]
Solve the equation

- **Equation of SHM**

\[ m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos(\omega_d t) \]

- **Steady state solution:**

\[ x(t) = x_m \cos(\omega_d t + \phi) \]

where

\[ x_m = \frac{F_0}{\sqrt{m^2 (\omega_d^2 - \omega^2)^2 + b^2 \omega_d^2}} \]
Forced Harmonic Motion and Resonance

\[ x_m = \frac{F_0}{\sqrt{m^2 (\omega_d^2 - \omega^2)^2 + b^2 \omega_d^2}} \]

\[ \omega_{max}^2 = \omega^2 - \frac{1}{2} \frac{b^2}{m^2} \]

*Resonance*: the amplitude is peaked when the driving frequency \( \omega_d \) nears the natural frequency \( \omega \)
Two Coupled Oscillators

- Dynamics of mass 1 (displacement $x_1(t)$)
  \[ m \frac{d^2 x_1}{dt^2} = -k x_1 - k(x_1 - x_2) \]

- Dynamics of mass 2 (displacement $x_2(t)$)
  \[ m \frac{d^2 x_2}{dt^2} = -k x_2 + k(x_1 - x_2) \]
Solve the differential equations

- **Solution:**

\[ x_1(t) = \frac{1}{2} x_{m1} \cos(\omega_1 t + \phi_1) + \frac{1}{2} x_{m2} \cos(\omega_2 t + \phi_2) \]
\[ x_2(t) = \frac{1}{2} x_{m1} \cos(\omega_1 t + \phi_1) - \frac{1}{2} x_{m2} \cos(\omega_2 t + \phi_2) \]

- **Note:** \( \omega_1 = \sqrt{\frac{k}{m}}, \ \omega_2 = \sqrt{\frac{3k}{m}} \)

\( x_{m1}, x_{m2}, \phi_1, \phi_2 \): determined by initial conditions
Energy in Coupled Oscillator

- Potential energy:
  \[ U(t) = \frac{1}{2} kx_1^2 + \frac{1}{2} k(x_2 - x_1)^2 + \frac{1}{2} kx_2^2 \]

- Kinetic energy:
  \[ K(t) = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2 \]

- Total energy:
  \[ E(t) = U(t) + K(t) = \frac{1}{4} kx_{m1}^2 + \frac{3}{4} kx_{m2}^2 \]
Forced Coupled Oscillators

- Dynamics of mass 1 (displacement $x_1(t)$)
  \[ m \frac{d^2 x_1}{dt^2} = -kx_1 - k(x_1 - x_2) + F_0 \cos(\omega_d t) \]

- Dynamics of mass 2 (displacement $x_2(t)$)
  \[ m \frac{d^2 x_2}{dt^2} = -kx_2 + k(x_1 - x_2) \]
Solve the differential equations

**Solution:**

\[
x_1(t) = x_{m1} \cos (\omega_d t)
\]

\[
x_2(t) = x_{m2} \cos (\omega_d t)
\]

where

\[
x_{m1}(\omega_d) = \frac{F_0}{m} \frac{2\omega_1^2 - \omega_d^2}{(\omega_1^2 - \omega_d^2)(\omega_2^2 - \omega_d^2)}
\]

\[
x_{m2}(\omega_d) = \frac{F_0}{m} \frac{\omega_1^2}{(\omega_1^2 - \omega_d^2)(\omega_2^2 - \omega_d^2)}
\]
Many Coupled Oscillators

- $N$ modes, $N$ resonances
- Motions travels from one side to the other
- $N \to \infty$ Wave
The Electrical Mechanical Analogy

$q$ corresponds to $x$, \hspace{1cm} 1/C corresponds to $k$,  
i corresponds to $v$, \hspace{1cm} and \hspace{1cm} $L$ corresponds to $m$.

<table>
<thead>
<tr>
<th>Block–Spring System</th>
<th>$L$ Oscillator</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Element</strong></td>
<td><strong>Energy</strong></td>
</tr>
<tr>
<td>Spring</td>
<td>Potential, $\frac{1}{2}kx^2$</td>
</tr>
<tr>
<td>Block</td>
<td>Kinetic, $\frac{1}{2}mv^2$</td>
</tr>
<tr>
<td>$v = dx/dt$</td>
<td></td>
</tr>
</tbody>
</table>
Kirchhoff's Voltage Law

\[
\frac{q}{C} + L \frac{di}{dt} = 0
\]

Capacitor: \( i = \frac{dq}{dt} \)

Equation of LC oscillation?
Equation of SHM
\[ \frac{d^2 q}{dt^2} + \frac{1}{LC} q = 0 \]

Solution: \( q(t) = Q \cos(\omega t + \phi) \)

where
\[ \omega = \sqrt{\frac{1}{LC}} \]

\( Q \) and \( \phi \) are determined by the initial conditions.
Energy in LC Oscillator

- **Electrical energy:**
  \[
  U_E(t) = \frac{q^2}{2C} = \frac{Q^2}{2C} \cos^2(\omega t + \phi)
  \]

- **Magnetic energy:**
  \[
  U_B(t) = \frac{1}{2} Li^2 = \frac{1}{2} L \omega^2 Q^2 \sin^2(\omega t + \phi)
  \]
  \[
  = \frac{Q^2}{2C} \sin^2(\omega t + \phi)
  \]

- **Total energy:**
  \[
  E(t) = U_E(t) + U_B(t) = \frac{Q^2}{2C}
  \]
Damped Oscillations in an RLC Circuit

- Kirchhoff's Voltage Law
  \[
  \frac{q}{C} + iR + L \frac{di}{dt} = 0
  \]

- Capacitor: \( i = \frac{dq}{dt} \)

- Equation of RLC oscillation?
Damped Oscillations in an RLC Circuit

- **Equation of RLC**
  \[
  L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0
  \]

- **Solution:** \( q(t) = Q e^{-Rt/2L} \cos(\omega' t + \phi) \)

  where

  \[
  \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}
  \]

  \( Q \) and \( \phi \) are determined by the initial conditions.
Damped Oscillations in an RLC Circuit

- \( q(t) = Qe^{-\frac{Rt}{2L}}\cos(\omega't + \phi) \)

\[
\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}
\]

- \( R = 0 \Rightarrow \omega' = \omega = \sqrt{\frac{1}{LC}} \)

- \( R \ll \sqrt{\frac{L}{C}} \Rightarrow \omega' \approx \omega \)

- Increasing \( R \)?
Energy in RLC Oscillator

- **Electrical energy:**
  \[ U_E(t) = \frac{q^2}{2C} = \frac{Q^2}{2C} e^{-Rt/L} \cos^2(\omega't + \phi) \]

- **Magnetic energy:**
  \[ U_B(t) = \frac{1}{2} Li^2 \approx \frac{1}{2} L\omega'^2 Q^2 e^{-Rt/L} \sin^2(\omega't + \phi) \]
  \[ \approx \frac{Q^2}{2C} e^{-Rt/L} \sin^2(\omega't + \phi) \]

- **Total energy:**
  \[ E(t) = U_E(t) + U_B(t) \approx \frac{Q^2}{2C} e^{-Rt/L} \]
Forced Oscillation in an RLC circuit

- Kirchhoff's Voltage Law

\[ \frac{q}{C} + iR + L \frac{di}{dt} = \mathcal{E}_m \sin(\omega_d t) \]

- Capacitor: \[ i = \frac{dq}{dt} \]

- Equation of forced RLC oscillation?
Forced Oscillation in an RLC circuit

- Equation of forced RLC oscillation
  \[ L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \omega_d \varepsilon_m \cos(\omega_d t) \]

- Steady state solution:
  \[ i(t) = I \cos(\omega_d t + \phi) \]

where

\[ I = \frac{\varepsilon_m}{\sqrt{L^2 \left( \omega_d - \frac{\omega^2}{\omega_d} \right)^2 + R^2}} \]
Forced Oscillation and Resonance

\[ I = \frac{E_m}{\sqrt{L^2 \left( \omega_d - \frac{\omega^2}{\omega_d} \right)^2 + R^2}} \]

\[ \omega_{\text{max}}^2 = \omega^2 \]

**Resonance**: the current amplitude is peaked when the driving frequency \( \omega_d \) equals the natural frequency \( \omega \)
Other Types of SHOs

The figure shows a mass-spring system with the following components:

- A mass $m$ attached to a spring.
- The spring constant is $-k$.
- The displacement from equilibrium is $x_0$.
- The force acting on the mass is $F = -kx_0$.
- The mass is subject to gravity $mg$.
- The displacement $x$ is measured from a reference point.

Additionally, there is a schematic diagram of an electrical circuit with:

- Inductors $L$ in parallel.
- Capacitors $C, C'$.
- Currents $I_1, I_2, I_3$.
At time $t$, the displacement $y$ of at position $x$ is

$$y(x, t) = y_m \sin(kx \pm \omega t + \phi)$$

Answer: Wave travelling direction positive (negative) then $- (+)$. 
The Speed of Travelling Wave

A wave of a general shape

\[ y(x, t) = h(kx \pm \omega t) \]

What is the speed of this wave?

Answer:

\[ kx - \omega t = k(x + \Delta x) - \omega(t + \Delta t) \Rightarrow \]

\[ k \frac{dx}{dt} - \omega = 0 \Rightarrow v = \frac{dx}{dt} = \frac{\omega}{k} \]
Wave Speed on a Stretched String

- Vertical component of the force on the element $\Delta l$

$$\tau \frac{\Delta l}{R} = (\mu \Delta l) \frac{v^2}{R}$$

- The wave speed

$$v = \sqrt{\frac{\tau}{\mu}}$$
Energy Transmission on a Stretched String

- The power delivered by wave = The power delivered to the string

\[ P = \mathbf{F} \cdot \mathbf{v}_{\text{transverse}} = -\tau \frac{\partial y \partial y}{\partial x \partial t} \]

- For a harmonic travelling wave

\[ y(x, t) = y_m \sin(kx - \omega t) \]

We have

\[ P = \mu v \omega^2 y_m^2 \cos^2(kx - \omega t) \]

- Average power delivered

\[ P_{\text{avg}} = \frac{1}{2} \mu v \omega^2 y_m^2 \]
This leads to

\[ \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2} \]

Note \( \nu = \sqrt{\tau/\mu} \Rightarrow \text{WAVE EQUATION} \)

\[ \frac{\partial^2 y}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 y}{\partial t^2} \]

General differential equation governing the travel wave of ALL types.
Interference of Waves

Two waves: *same* amplitude, *same* wavelength, *same* direction

\[
y_1(x, t) = y_m \sin(kx - \omega t)
\]
\[
y_2(x, t) = y_m \sin(kx - \omega t + \phi)
\]

**What is the resultant wave?**

\[
y'(x, t) = \left[2y_m \cos \frac{1}{2} \phi \right] \sin \left(kx - \omega t + \frac{1}{2} \phi \right)
\]

**Special cases of \(\phi\)?**
Standing Waves

Two waves: \textit{same} amplitude, \textit{same} wavelength, \textit{opposite} directions

\[
y_1(x, t) = y_m \sin(kx - \omega t)
\]
\[
y_2(x, t) = y_m \sin(kx + \omega t)
\]

\textbf{The resultant wave:}

\[
y'(x, t) = [2y_m \sin(kx)] \cos\omega t
\]

\textbf{Nodes}: \sin kx = 0 \Rightarrow x = n \frac{\lambda}{2}, \quad n = 0, 1, 2 \ldots

\textbf{Antinodes}: \sin kx = \pm 1 \Rightarrow x = (n + \frac{1}{2}) \frac{\lambda}{2}, n = 0, 1, 2 \ldots
Standing Waves and Resonance

Wavelength

\[ L = n \frac{\lambda}{2} \Rightarrow \lambda = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots \]

Frequency

\[ f = \frac{v}{\lambda} = n \frac{v}{2L} = \frac{n}{2L} \sqrt{\frac{\tau}{\mu}}, \quad n = 1, 2, 3, \ldots \]

Harmonic series: \( f_1, f_2, f_3, \ldots \)

Harmonic number: \( n \)
Maxwell’s Rainbow

Fig. 33-1 The electromagnetic spectrum.
### Maxwell’s Equations

#### Table 32-1

<table>
<thead>
<tr>
<th>Name</th>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’ law for electricity</td>
<td>$\oint \vec{E} \cdot d\vec{A} = q_{enc}/\varepsilon_0$</td>
<td>Relates net electric flux to net enclosed electric charge</td>
</tr>
<tr>
<td>Gauss’ law for magnetism</td>
<td>$\oint \vec{B} \cdot d\vec{A} = 0$</td>
<td>Relates net magnetic flux to net enclosed magnetic charge</td>
</tr>
<tr>
<td>Faraday’s law</td>
<td>$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$</td>
<td>Relates induced electric field to changing magnetic flux</td>
</tr>
<tr>
<td>Ampere–Maxwell law</td>
<td>$\oint \vec{B} \cdot d\vec{s} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0i_{enc}$</td>
<td>Relates induced magnetic field to changing electric flux and to current</td>
</tr>
</tbody>
</table>

*Written on the assumption that no dielectric or magnetic materials are present.*
Fig. 33-3  An arrangement for generating a traveling electromagnetic wave in the shortwave radio region of the spectrum: an $LC$ oscillator produces a sinusoidal current in the antenna, which generates the wave. $P$ is a distant point at which a detector can monitor the wave traveling past it.
Traveling EM Wave: Qualitatively

\[ \frac{\partial^2 E}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}, \quad \frac{\partial^2 B}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2} \]

Wave equation!!!
What is the wave speed?

\[ \frac{\partial^2 E}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}, \quad \frac{\partial^2 B}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2} \]

- Travelling wave speed

\[ v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]

- In vacuum

\( \mu_0 = 1.257 \times 10^{-6} \text{ H/m}, \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \)

- Travelling wave speed

\[ v = c = 2.998 \times 10^8 \text{ m/s} \]
Traveling EM Wave: Qualitatively

\[
\frac{\partial^2 E}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}, \quad \frac{\partial^2 B}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}
\]

\[
B = \frac{1}{c} E_m \sin(kx - \omega t + \phi) = \frac{1}{c} E
\]
Oscillations and Waves
Final Revision

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